Brief description of the main present research interests of Umberto Zannier

Specialization theorems

They concern objects moving in a parameterised family. Usualy one tries to prove that certain 'generic' properties still hold after (some) specialisations of the parameters.

This general theme traces back to long ago, well-known instances occurring with *Bertini irreducibility theorems.*

`Hilbert Property`. An arithmetical analogue of Bertini's theorem is *Hilbert Irreducibility theorem*, of diophantine nature so to say, and useful e.g. in the Inverse Galois Problem. Recently the so-called *Hilbert Property* (introduced by Serre) has been studied for varieties other than $\mathbf{P}_n$ (which corresponds to the original Hilbert theorem). For example it has been found that the topology of the variety is highly relevant, and the property has been proved or disproved as the case may be, for certain non-rational varieties including some Fermat quartic surfaces in $\mathbf{P}_3$ (like $x^4 + y^4 = z^4 + w^4$) and their quotients.

More recently, specialization principles reappeared also in new shapes:

`Algebraic families of abelian varieties`.

The best known example is probably the *Legendre elliptic family*, defined by the equation $y^2 = x(x-1)(x-\lambda)$ where λ is the parameter describing the family, this time of elliptic curves (for $\lambda \neq 0, 1$). Note that we have an algebraic group law on every member of the family.

- Some issues arise when we have *sections* of the family, which may be viewed as points of the generic member. For instance take $\sigma = (x, y)$ where $x = x(\lambda) = \lambda + 1$, $y(\lambda) = \sqrt{(\lambda+1)\lambda}$.

A natural question is to ask when sections which are non-torsion (with respect to the group law) become torsion (i.e. $n \cdot \sigma = 0$ for some $n > 0$) when specialised. Much work has been done on this problem, going back to Manin-Demianenko and Silverman-Tate (proving *bounded height*) until recently, when *finiteness* has been proved in certain cases. These last theorems are special cases of the *Pink-Zilber conjecture* which vastly generalised e.g. the *Manin-Mumford conjecture.*

`Existence of abelian varieties over` $\overline{\mathbf{Q}}$ `with 'generic' properties`

The space of abelian varieties of given dimension g may be 'parametrized' by the points of an algebraic variety, denoted $\mathcal{A}_g$. It is a question of the above type to ask whether specialisation to points in $\mathcal{A}_g$ with algebraic coordinates may preserve some properties which are true generically. Instances of this are questions by N. Katz-Oort, treated and solved by Chai-Oort and Tsimerman. But one can requires certain other generic property, and an answer has been given in recent work.

`Betti map`

The above issues also lead to the study of the so-called *Betti map* of the section, considered implicitly already by Manin in the 1960s. This map is real-analytic but also has intricate complex-analytic properties, and is important e.g. for the distribution of torsion-values of a section; it recently appeared in other issues as well, with applications to Chow groups (by Voisin). Recent work yields new information and new links, e.g. with the theory of *functional transcendence.*

`Integration of differentials in elementary terms`

The above subject has shown connection with old problems regarding *integration in elementary terms* of algebraic differentials (a topic going back to Abel). In the classical sense one seeks to integrate a differential using only algebraic operations, or operations of exponential or logarithmic type. It was conjectured that a parametric family of differentials either are identically likewise integrable or this may be done for only finitely many specialisations. Now, recently it has been proved that this is not generally true, however giving also a general positive results which allows to decide what happens in any given situation.

`Pell equations in polynomials`

Hyperelliptic curves and differential lead to the so-called *Pell equation* $X^2 - DY^2 = 1$: this is well known in the case of integers (going back to antiquity and rediscovered by Fermat). But one may also study a polynomial variant, as done by Abel in 1926.

Such equation appears in many mathematical issues of different kind, and in particular is linked to the topics recalled above. One may study also the associated continued fractions, of functions of the shape $\sqrt{D(t)}$. In the classical case of real quadratic surds $\sqrt{D}$, $D \in \mathbf{N}$, studied by Lagrange and Galois, the continued fraction is always periodic. In the new context already Abel realized this does not always hold. Recently it has been found that however the sequence of degrees of the *partial quotients* is always periodic.

`Families of diophantine equations`

Diophantine equations depending on parameters have often appeared in the literature. Just to quote an example, consider $\mathcal{E}_t : X^3 - (t^3 + 1)Y^3 = 1$, to be solved in integers x, y, once the integer t has been fixed. This is a family of *Thue's equations*; it has long been known that for each $t \in \mathbf{Z}$ there are only the integer solutions $x = -t, y = -1$ or $x = -1, y = 0$.

Now, it is standard that these equations may be translated into diophantine problems of *multiplicative type*. In recent work general such problems have been studied, proving *bounded height* for the t which yield 'unexpected' solutions, which applies to yield results like the above. This also raises natural open questions, for instance in the realm of abelian varieties.

Integral points over number fields and function fields, transcendence

The last quoted theme brings us into the topic of diophantine equations. Among the fairly recent research, some results have been derived from the Schmidt Subspace Theorem, a far reaching extension of Roth's theorem in Diophantine Approximation. Especially, on applying this theorem to convenient embeddings in affine spaces, one may obtain several results on *multiplicative diophantine equations*. The Subspace Theorem, applied in suitable ways, also implies conclusions on the *transcendence* of certain numbers.

One may also study equations in polynomials in place of integers, which is a geometric version of these problems. Recently it has been found that this has links e.g. with the theory of *fewnomials*, that is, polynomials with a bounded number of terms. For instance, an old conjecture of Erdös and Rény, solved by Schinzel, predicted that: *If the square of a polynomial has a bounded number of terms, the same is true for the polynomial.* The said methods allow in particular to prove broad extensions of this, which escape from previously known techniques.

The alluded works have been done in various collaborations with F. Amoroso, Y. André, E. Bombieri, P. Corvaja, C. Fuchs, Z. Gao, V. Mantova, D. Masser, J. Pila.