Characterization of oscillations and mass concentration occurring along sequences of $\mathscr{A}$-free functions

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Nonlinear analysis for continuum mechanics

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Set up: The PDE constraint

We consider a homogeneous linear system of PDEs

$$\mathscr{A}u = \sum_{|\alpha|=k} A_\alpha \partial^\alpha u, \qquad u : \mathbb{R}^d \to \mathbb{R}^M,$$

where

- the coefficients are constant: $A_\alpha \in \mathrm{Lin}(\mathbb{R}^M; \mathbb{R}^N)$
- ∂^α denotes the $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ distributional derivative

Main assumption: The principal symbol

$$\mathbb{A}(\xi) := \sum_{|\alpha|=k} A_\alpha \xi^\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}, \quad \xi \in \mathbb{R}^d.$$

satisfies the **constant-rank condition**

$$\dim(\mathrm{Im}\, \mathbb{A}(\xi)) = r \in \mathbb{N}_0 \qquad \text{for all } \xi \neq 0.$$

Minimization of integral energies

$$I_f(u) := \int_\Omega f(u(x))\,\mathrm{d}x,$$

where $f : \mathbb{R}^M \to \mathbb{R}$ has linear growth

$$|f(z)| \eqsim C(1 + |z|) \quad z \in \mathbb{R}^M,$$

defined on configurations

$$u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^M$$

satisfying

$$\mathscr{A}u = 0 \quad \text{on } \Omega \qquad (\mathscr{A}\text{-free})$$

Caveats:

- minimizing sequences u_j are only L^1-bounded:

$$u_j \overset{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\Omega; \mathbb{R}^M)$$

  ⤳ fine oscillations + appearance of mass concentration

- Need for relaxation: extend the set of configurations

$$u \in L^1(\Omega; \mathbb{R}^M) \cap \ker \mathscr{A} \longrightarrow \mu \in \mathscr{M}(\Omega; \mathbb{R}^M) \cap \ker \mathscr{A}$$

  ⤳ extend the functional

$$\overline{I_f}(\mu) = \int_\Omega f(\mu^a(x))\, dx + \int_\Omega f^\infty \left(\frac{d\mu}{d|\mu|}(x) \right) d|\mu^s|(x),$$

- compensated compactness kicks in:

$$\mathscr{A} u_j = 0, \quad \mathscr{A} \mu = 0$$

  ⤳ certain oscillations/concentrations are prevented

- There are bad directions, such as the the wave-cone directions

$$\Lambda_{\mathscr{A}} := \bigcup_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}(\xi) \subset \mathbb{R}^M,$$

where oscillations and concentrations are compatible with the PDE
$\rightsquigarrow$ existence of solutions requires a type of convexity:

$$f \text{ is } \mathscr{A}\text{-quasiconvex} \quad \text{(throw it under the rug... for now)}$$

Motto: Characterize,

by testing with $\mathscr{A}$-quasiconvex integrands,

all possible ways in which

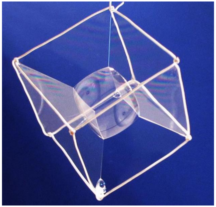
$$u_j \overset{*}{\rightharpoonup} \mu, \qquad \mathscr{A} u_j = 0,$$

may develop oscillations/concentrations.

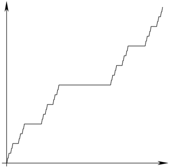
Curl-free fields (BV-theory)

$$\mathbb{R}^M = \mathbb{R}^{m\times d}$$

$$u = Dw \ \ (w : \mathbb{R}^d \to \mathbb{R}^m) \quad \Leftrightarrow \quad \operatorname{curl} u = 0$$



- geometric problems
- memory alloys
- sessile drops
- optimal design for linear conductivity and linear plate models

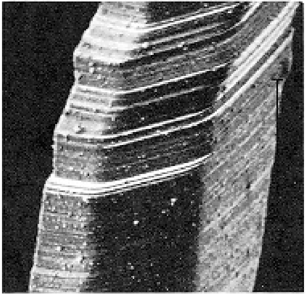


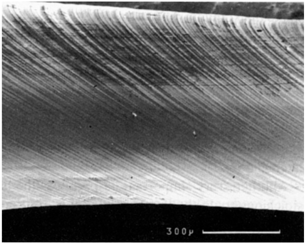
Saint Venant conditions (BD-theory)

$$\mathbb{R}^M = \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

$$u = \frac{Dw + Dw^T}{2} \quad (w : \mathbb{R}^d \to \mathbb{R}^d) \quad \Leftrightarrow \quad L(D^2)u = 0$$

- linear elasticity
- perfect plasticity



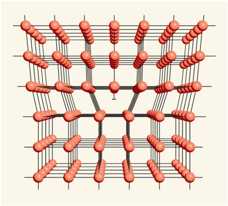


Divergence-free systems and m-currents without boundary

$$\mathbb{R}^M \cong \mathbb{R}^{k\times d}, \bigwedge_m \mathbb{R}^d$$

$$\nabla\cdot M = \begin{pmatrix} \nabla\cdot M^1 \\ \vdots \\ \nabla\cdot M^k \end{pmatrix} = 0, \quad \partial T = 0$$

- optimal design with vanishing volume
- dislocations



Lifting the rug a bit...

Definition 1 (Morrey '66, Dacorogna '82, Fonseca–Müller '99)

A function $f : \mathbb{R}^M \to \mathbb{R}$ is called **$\mathscr{A}$-quasiconvex** if

$$f(z) \leq \int_{\mathbb{T}^d} f(z + w(y))\,\mathrm{d}y, \quad z \in \mathbb{R}^M,$$

for all $w \in \mathrm{C}_c^\infty([0,1]^d, \mathbb{R}^M)$ with $\mathscr{A}w = 0$.

– *"Jensen's inequality along $\mathscr{A}$-free fields"*

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Notion of Young measure (oscillations)

Let $u_j \rightharpoonup u$ in $L^1(\Omega)$ (equi-integrable) and consider a parameterized family

$$\boldsymbol{\nu} = \{\nu_x\}_{x \in \Omega} \subset \mathrm{Prob}(\mathbb{R}^M).$$

We say that "u_j generates $\boldsymbol{\nu}$" provided that

$$\nu_x(A) = \text{"Probability}\{u_j(y) \in A : j \to \infty, y \sim x\}\text{"} \qquad A \subset \mathbb{R}^M,$$

Theorem 2 (Oscillations; Fonseca–Müller '99)

A family $\boldsymbol{\nu} = \{\nu_x\}_\Omega \subset \mathrm{Prob}(\mathbb{R}^M)$ *is generated by a sequence of equi-integrable* $\mathscr{A}$***-free functions*** *if and only if*

(i) *there exists* $u \in \mathrm{L}^1(\Omega; W)$ *such that* $\mathscr{A}u = 0$ *and*

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(ii) *for all linear growth* $\mathscr{A}$***-qc*** *integrands* $f : \mathbb{R}^M \to \mathbb{R}$ *it holds*

$$f(\mathbb{E}(\nu_x)) \leq \int_{\mathbb{R}^M} f(z)\,\mathrm{d}\nu_x(z) \quad \text{for a.e. } x \in \Omega.$$

— "$\boldsymbol{\nu} = \{\nu_x\}_{x\in\Omega}$ *are the* ***probability distributions*** *of a sequence of equi-integrable* $\mathscr{A}$*-free functions if and only if (i) and (ii) are satisfied*"

Definition 3

A parametrized triple $\boldsymbol{\nu} = (\nu_x, \lambda, \nu_x)_{x\in\overline{\Omega}}$,

- $\nu_x \in \mathrm{Prob}(\mathbb{R}^M)$
- $\lambda \in \mathscr{M}(\overline{\Omega})$
- $\nu_x^\infty \in \mathrm{Prob}(\mathbb{S}^{M-1})$

is called a **generalized Young measure**

We say that "u_j generates $\boldsymbol{\nu}$ if"

- ν_x is the oscillation probability distribution of u_j about x
- $\lambda(dx)$ is the quantity of mass carried by $|u_j|$ towards x
- ν_x^∞ is the angular distribution of $\frac{u_j}{|u_j|}$ about x

Recent theory: Generalized Young measures

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Decompose

$$\lambda = \lambda^a \mathscr{L}^d + \lambda^s, \qquad \lambda^s \perp \mathscr{L}^d,$$

then (formally)

- $\operatorname{supp} \lambda^a$ is the of points with diffuse concentrations (Farkir's carpet)
- $\operatorname{supp} \lambda^s$ is the set of points with dirac mass-like pure concentrations

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(i) *there exists* $\mu \in \mathscr{M}(\Omega; \mathbb{R}^M)$ *such that* $\mathscr{A}\mu = 0$ *and*

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(ii) *for all linear growth* $\mathscr{A}$*-qc* $f : \mathbb{R}^M \to \mathbb{R}$ *integrands,*

$$f(\,\mathbb{E}(\nu_x) + \lambda^a(x)\mathbb{E}(\nu_x^\infty)\,) \leq \langle f, \nu_x \rangle + \lambda^a(x)\langle f^\infty, \nu_x^\infty \rangle$$

for almost every $x \in \Omega$*,*

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(iii) *the angular concentration part satisfies (often trivial)*

$$\mathrm{supp}(\nu_x^\infty) \subset \mathrm{span}\{\Lambda_{\mathscr{A}}\} \quad \lambda^s\text{-}a.e., \qquad \Lambda_{\mathscr{A}} := \bigcup_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}(\xi).$$

* After my proof appeared, Kristensen and Raita have proposed an interesting alternative proof for a slightly less general version of this theorem.

Comment: Similar proof gives a characterization of all Young measures generated by $\mathscr{B}$-gradients, i.e.,

$$u_j = \mathscr{B}w_j \text{ generates } \boldsymbol{\nu}.$$

Previous work

- Kristensen and Rindler '11 ($u = Dw$, $\mathscr{A} = \text{curl}$)
- Baía, Matias and Santos '13 ($\mathscr{A}$ is 1st-order + extra ass.)
- Rindler and De Philippis '17 ($u = Ew$, $\mathscr{A} = \text{St. Venant}$)

Rigidity for elliptic systems (Müller '95, Sverak '91, ...). Let $L \leq \mathbb{R}^M$ be space of with no $\Lambda_{\mathscr{A}}$-connections, i.e.,

$$L \cap \Lambda_{\mathscr{A}} = \{0\}.$$

If $\{\mu_j\} \subset L^1(\Omega; \mathbb{R}^M)$ is a sequence of $\mathscr{A}$-free functions satisfying

$$\begin{aligned} \mu_j \, \mathscr{L}^d &\overset{*}{\rightharpoonup} 0 \quad \text{in } \mathscr{M}(\Omega), \\ \operatorname{dist}(\mu_j, L) &\to 0 \quad \text{in measure}, \end{aligned}$$

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then

$$\mu_j \to 0 \text{ in measure.}$$

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Essentially: L^1_w-compactness due to the existence of (L^1, L^1_w)-multipliers.

Q: Can we expect more to prevent concentrations (equi-integrability) provided that we require the stronger condition

$$\operatorname{dist}(\mu_j, L) \to 0 \quad \text{in } L^1(\Omega)\,?$$

A: It fails... big time!

Lemma 5 (A-R)

Let

$$L \leq \operatorname{span} \Lambda_{\mathscr{A}}$$

be a non-trivial subspace and assume that L *has no non-trivial* $\Lambda_{\mathscr{A}}$*-connections, i.e.,*

$$L \cap \Lambda_{\mathscr{A}} = \{0\}.$$

Then, there exists a sequence $\{w_j\} \subset \mathrm{C}^\infty(\Omega; \mathbb{R}^M)$ *of* $\mathscr{A}$*-free measures satisfying*

$$\begin{aligned} \min\{w_j, R\} &\to 0 \quad \text{in } L^1(\Omega; \mathbb{R}^M) \quad \forall R > 0 \\ w_j \mathscr{L}^d &\overset{*}{\rightharpoonup} 0 \quad \text{in } \mathscr{M}(\Omega; \mathbb{R}^M), \\ \operatorname{dist}(w_j, L) &\to 0 \quad \text{in } \mathrm{L}^1(\Omega), \end{aligned}$$

but

$$w_j \not\rightharpoonup 0 \in \mathrm{L}^1(\Omega; \mathbb{R}^M), \text{ and}$$

$$\{|w_j|\} \text{ is not locally equi-integrable on } \textbf{any} \text{ sub-domain of } \Omega.$$

Extending Ball–James rigidity: De Philippis, Palmieri and Rindler '18 showed that if

$$A - B \notin \Lambda_{\mathscr{A}}$$

and $\{v_j\} \subset \mathrm{L}^1(\Omega; \mathbb{R}^M)$ is a sequence of $\mathscr{A}$-free functions satisfying

$$\mathrm{dist}(v_j, \{A, B\}) \to 0 \text{ in } \mathrm{L}^1(\Omega),$$

then, up to extracting a subsequence,

$$v_j \to const. \text{ in } \mathrm{L}^1(\Omega).$$

Q: What happens if we allow for concentrations, i.e.,

$$\mathrm{dist}(v_j, \{A, B\}) \to 0 \text{ in measure ?}$$

Lemma 6 (A-R)

Let $A_1, \dots, A_n \in (\text{span}\{\Lambda_{\mathscr{A}}\} \cap S_W)$ *be unit vectors satisfying*

$$0 \in \text{convex hull}\{A_1, \dots, A_n\}.$$

There exists a sequence of $\mathscr{A}$*-free functions* $\{w_j\} \in \mathrm{C}^\infty(\Omega; \mathbb{R}^M)$ *generating the triple*

$$(\delta_{A_1}, \mathscr{L}^d \llcorner \Omega, p),$$

where

$$p = c_1 \delta_{A_1} + \dots + c_n \delta_{A_n},$$

$$c_1, \dots, c_n \in [0, 1] \quad \textit{and} \quad c_1 A_1 + \dots c_n A_n = 0.$$

Corollary 7

Let $A \notin \Lambda_{\mathscr{A}}$*. There exists a sequence of* $\mathscr{A}$*-free measures* $\{v_j\} \subset \mathrm{C}^\infty(\Omega; \mathbb{R}^M)$ *satisfying (A and* $-A$ *are not* $\Lambda_{\mathscr{A}}$*-connected)*

$$\text{dist}(v_j, \{A, -A\}) \to 0 \textit{ in measure},$$

but $\{|v_j|\}$ *is not equi-integrable on* ***any*** *sub-domain of* Ω*.*

Najlepša hvala!