The 8th European Congress of Mathematics

Conditions of global solvability, Lyapunov stability, Lagrange stability and dissipativity for time-varying semilinear differential-algebraic equations, and applications

Maria Filipkovska

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine

Consider **implicit ordinary differential equations** (ODEs) of the form

$$\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \qquad t \in [t_+,\infty), \tag{1}$$

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) = f(t,x(t)), \tag{2}$$

where $t_+ \geq 0$, $A, B\colon [t_+,\infty) \to L(\mathbb{R}^n)$, $f\colon [t_+,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

The time-varying operators $A(t)$, $B(t)$ can be degenerate.

The differential equations (DEs) (1) and (2) with a degenerate (for some t) operator $A(t)$ are called **time-varying (nonautonomous) degenerate DEs** or **time-varying differential-algebraic equations (DAEs)**. In the terminology of DAEs, equations of the form (1), (2) are commonly referred to as **semilinear**.

We study the initial value problem (the Cauchy problem) for the DAEs (1), (2) with the initial condition

$$x(t_0) = x_0. \tag{3}$$

It is assumed that the characteristic operator pencil $\lambda A(t) + B(t)$ ($\lambda \in \mathbb{C}$ is a parameter), associated with the linear part of the DAE (1) or (2), is a regular pencil of index not higher than 1.

Semilinear DAEs of the type (1) include **semi-explicit DAEs**

$$\begin{aligned} &\dot{y} = h(t,y,z), \\ &0 = g(t,y,z), \qquad h\colon [t_+,\infty)\times\mathbb{R}^{k+m}\to\mathbb{R}^k,\ g\colon [t_+,\infty)\times\mathbb{R}^{k+m}\to\mathbb{R}^m, \end{aligned}$$

and **Hessenberg DAE**

$$\begin{aligned} &\dot{y} = h(t,y,z), \\ &0 = g(t,y), \qquad h\colon [t_+,\infty)\times\mathbb{R}^{k+m}\to\mathbb{R}^k,\ g\colon [t_+,\infty)\times\mathbb{R}^{k}\to\mathbb{R}^m. \end{aligned}$$

DAEs or degenerate DEs are also called **descriptor systems**, **algebraic-differential equations** and **differential equations (or dynamical systems) on manifolds**.

The development of the theory of differential-algebraic equations

(degenerate differential equations, descriptor systems, algebraic-differential systems, singular systems)

The solvability: K. Weierstrass (1867), L. Kronecker (1890), V.P. Skripnik (1964), A.G. Rutkas (1975), R.E. Showalter (1975), S.L. Campbell (1976), Yu.E. Boyarintsev (1977), A. Favini (1977), V.F. Chistyakov (1980), L.R. Petzold (1982), L.A. Vlasenko (1987), E. Hairer (1988), P. Kunkel (1991), V. Mehrmann (1991), V.P. Yakovecz (1991), R. März (1994), C. Tischendorf (1994), A.A. Shcheglova (1995), A.M. Samoilenko (2000), R. Riaza (2000), Yu.E. Gliklikh (2014) and others.

The stability: L. Dai (1989), R. März (1994), C. Tischendorf (1994), A.A. Shcheglova (2004), V.F. Chistyakov (2004), Yu.E. Boyarintsev (2006), S.L. Campbell (2009), V.H. Linh (2009), Sh. Xu, J. Lam (2006), T. Berger, A. Ilchmann (2010) and others.

Numerical methods: Gear C.W. (1971), L.R. Petzold (1983), E. Hairer, Ch. Lubich, M. Roche (1988), Yu.E. Boyarintsev, V.A. Danilov, V.F. Chistyakov (1989), G. Wanner, U.M. Ascher (1991), P.J. Rabier, W.C. Rheinboldt (1994), G.Yu. Kulikov (1993), P. Benner, R. Byers, V. Mehrmann, D. Kressner and others.

Fields of application of the theory of DAEs are radioelectronics, control theory, cybernetics, mechanics, robotics technology, economics, ecology and chemical kinetics.

In particular, semilinear DAEs are used to describe

- transient processes in electrical circuits and the dynamics of neural networks (R. Riaza, A.G. Rutkas, L.A. Vlasenko, K.E. Brenan, S.L. Campbell, L.R. Petzold, R. März, C. Tischendorf and others),
- the dynamics of complex mechanical and technical systems (e.g., robots) (P.J. Rabier, W.C. Rheinboldt, B. Fox, L.S. Jennings, A.Y. Zomaya, B. Siciliano and others),
- the dynamics of various descriptor systems (R. Riaza, J. Zufiria, P. Kunkel, V. Mehrmann, J.C. Engwerda, I.E. Wijayanti and others),
- multi-sectoral economic models (M. Morishima, S.R. Khachatryan and others),
- kinetics of chemical reactions (L.V. Knaub, A.E. Novikov, E.A. Novikov).

Let for each $t \geq t_+$ the pencil $\lambda A(t) + B(t)$ be regular and let there exist functions $C_1 : [t_+,\infty) \to (0,\infty)$, $C_2 : [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$ the pencil resolvent $R(\lambda,t) = (\lambda A(t) + B(t))^{-1}$ satisfies the constraint

$$\|R(\lambda,t)\| \leq C_1(t), \quad |\lambda| \geq C_2(t). \tag{4}$$

Then for each $t \in [t_+,\infty)$ there exist the two pairs of mutually complementary projectors

$$P_j(t) : \mathbb{R}^n \to X_j(t) \text{ and } Q_j(t) : \mathbb{R}^n \to Y_j(t), \ j = 1,2,$$

$(P_i(t)P_j(t) = \delta_{ij}P_i(t)$, $P_1(t) + P_2(t) = I_{\mathbb{R}^n}$, $Q_i(t)Q_j(t) = \delta_{ij}Q_i(t)$, $Q_1(t) + Q_2(t) =)$, which generate the direct decompositions of spaces

$$\mathbb{R}^n = X_1(t)\dot{+}X_2(t), \mathbb{R}^n = Y_1(t)\dot{+}Y_2(t), \text{ such that} \tag{5}$$

$$A(t) = \begin{pmatrix} A_1(t) & 0 \\ 0 & 0 \end{pmatrix}, B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} : X_1(t)\dot{+}X_2(t) \to Y_1(t)\dot{+}Y_2(t), \tag{6}$$

$X_2(t) = \operatorname{Ker} A(t)$, $Y_1(t) = A(t)\mathbb{R}^n$,

and there exist $A_1^{-1}(t)$ (if $X_1(t) \neq \{0\}$) and $B_2^{-1}(t)$ (if $X_2(t) \neq \{0\}$).

The auxiliary operator $G(t) = A(t) + B(t)P_2(t) \in L(\mathbb{R}^n)$, $G(t)X_j(t) = Y_j(t)$, has the inverse $G^{-1}(t) = A_1^{-1}(t)Q_1(t) + B_2^{-1}(t)Q_2(t) \in L(\mathbb{R}^n)$.

[Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

The condition (4) (a regular pencil $\lambda A(t) + B(t)$ has index not higher than 1) means that either the point $\mu = 0$ is a simple pole of the resolvent $(A(t) + \mu B(t))^{-1}$ (this is equivalent to the fact that $\lambda = \infty$ is a removable singular point of the resolvent $R(\lambda,t) = (\lambda A(t) + B(t))^{-1}$), or $\mu = 0$ is a regular point of the pencil $A(t) + \mu B(t)$ (i.e., there exists the resolvent $R(\lambda,t)$ at the point $\mu = 0$ and, hence, $A(t)$ is nondegenerate).

For each $t \in [t_+,\infty)$ the projectors can be constructively determined by the formulas

$$\begin{aligned} P_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} R(\lambda,t)\, A(t)\, d\lambda, \quad P_2(t) = I_{\mathbb{R}^n} - P_1(t), \\ Q_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} A(t)\, R(\lambda,t)\, d\lambda, \quad Q_2(t) = I_{\mathbb{R}^n} - Q_1(t). \end{aligned} \tag{7}$$

For each t any $x \in \mathbb{R}^n$ can be uniquely represented in the form

$$x = x_{p_1}(t) + x_{p_2}(t), \quad x_{p_i}(t) = P_i(t)x \in X_i(t).$$

The DAE (1) $[A(t)x(t)]' + B(t)x(t) = f(t,x(t))$ is reduced to the equivalent system

$$[P_1(t)x(t)]' = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]P_1(t)x(t) + G^{-1}(t)Q_1(t)f(t,x(t)),$$
$$G^{-1}(t)Q_2(t)[f(t,x(t)) - A'(t)P_1(t)x(t)] - P_2(t)x(t) = 0 \qquad \text{or}$$

$$x_{p_1}'(t) = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x), \qquad (8)$$
$$G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)] - x_{p_2}(t) = 0. \qquad (9)$$

$V'_{(8)}(t,x_{p_1}(t)) = \frac{\partial V}{\partial t}(t,x_{p_1}(t)) + \Big(\frac{\partial V}{\partial z}(t,x_{p_1}(t)), \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x_{p_1}(t) + x_{p_2}(t))\Big)$ is the derivative of the function $V(t,z)$ along the trajectories of the equation (8), where $V(t,z)$ is a continuously differentiable and positive definite scalar function.

Introduce the manifold

$$L_{t_+} = \{(t,x) \in [t_+,\infty) \times \mathbb{R}^n \mid Q_2(t)[B(t)x + A'(t)P_1(t)x - f(t,x)] = 0\}. \qquad (10)$$

The consistency condition $(t_0,x_0) \in L_{t_+}$ for the initial point (t_0,x_0) is one of the necessary conditions for the existence of a solution of the initial value problem (1), (3).

The IVP (1), (3): $\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \quad x(t_0) = x_0.$

Definitions

A solution $x(t)$ of the initial value problem (IVP) (1), (3) is called **global** or **defined in the future** if it exists on $[t_0,\infty)$.

A solution $x(t)$ of the IVP (1), (3) is called **Lagrange stable** if it is global and bounded, i.e., $\sup\limits_{t\in[t_0,\infty)} \|x(t)\| < \infty$.

A solution $x(t)$ of the IVP (1), (3) has a **finite escape time** (is **blow-up in finite time**) and is called **Lagrange unstable** if it exists on some finite interval $[t_0,T)$ and is unbounded, i.e., $\lim\limits_{t\to T-0} \|x(t)\| = \infty$.

The equation (1) is called **Lagrange stable** if every solution of the IVP (1), (3) is Lagrange stable (the DAE is Lagrange stable for every consistent initial point).

The equation (1) is called **Lagrange unstable** if every solution of the IVP (1), (3) is Lagrange unstable.

J. La Salle obtained the theorems on the global solvability, the Lagrange stability and instability of the explicit ODE $x' = f(t,x)$ [J. La Salle, S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, 1961].

Solutions of the equation (1) are called **ultimately bounded**, if there exists a constant $K > 0$ (K is independent of the choice of t_0, x_0) and for each solution $x(t)$ with an initial point (t_0, x_0) there exists a number $\tau = \tau(t_0, x_0) \geq t_0$ such that $\|x(t)\| < K$ for all $t \in [t_0 + \tau, \infty)$.

The equation (1) is called **ultimately bounded** or **dissipative**, if for any consistent initial point (t_0, x_0) there exists a global solution of the initial value problem (1), (3) and all solutions are ultimately bounded.

If the number τ does not depend on the choice of t_0, then the solutions of (1) are called *uniformly ultimately bounded* and the equation (1) is called *uniformly ultimately bounded* or *uniformly dissipative*.

Ultimately bounded systems of explicit ODEs $x' = f(t, x)$, which are also called dissipative systems and D-systems, were studied in [Yoshizawa T., *Stability theory by Liapunov's second method*, 1966] and [La Salle J., Lefschetz S., 1961].

The model of a radio engineering filter

A voltage source $e(t)$,
nonlinear resistances φ, φ_0, ψ,
a nonlinear conductance h,
a linear resistance r,
a linear conductance g,
an inductance L and
a capacitance C are given.

Let $e(t) \in C([0,\infty),\mathbb{R})$,
$\varphi(y), \varphi_0(y), \psi(y), h(y) \in C^1(\mathbb{R},\mathbb{R})$,
r, g, L, $C > 0$.

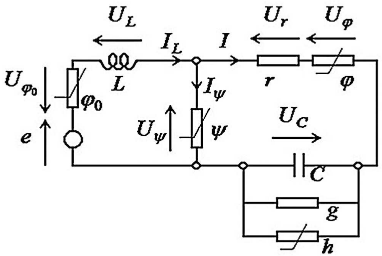

Fig. 1. The diagram of the electric circuit

The model of the circuit Fig. 1 is described by the system with the variables
$x_1 = I_L$, $x_2 = U_C$, $x_3 = I$:

$$L\frac{d}{dt}x_1 + x_2 + r x_3 = e(t) - \varphi_0(x_1) - \varphi(x_3), \quad (11)$$

$$C\frac{d}{dt}x_2 + g x_2 - x_3 = -h(x_2), \quad (12)$$

$$x_2 + r x_3 = \psi(x_1 - x_3) - \varphi(x_3). \quad (13)$$

$$A = \begin{pmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & r \\ 0 & g & -1 \\ 0 & 1 & r \end{pmatrix}$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \quad (14)$$

$$f(t,x) = \begin{pmatrix} e(t) - \varphi_0(x_1) - \varphi(x_3) \\ -h(x_2) \\ \psi(x_1 - x_3) - \varphi(x_3) \end{pmatrix}$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$

Lagrange stability of the model of a radio engineering filter. The particular cases.

$$\varphi_0(y) = \alpha_1 y^{2k-1}, \varphi(y) = \alpha_2 y^{2l-1}, \psi(y) = \alpha_3 y^{2j-1}, h(y) = \alpha_4 y^{2s-1}, \quad (15)$$
$$\varphi_0(y) = \alpha_1 y^{2k-1}, \varphi(y) = \alpha_2 \sin y, \psi(y) = \alpha_3 \sin y, h(y) = \alpha_4 \sin y, \quad (16)$$

$k,l,j,s \in \mathbb{N}$, $\alpha_i > 0$, $i = \overline{1,4}$, $y \in \mathbb{R}$.

For each initial point (t_0, x^0) satisfying $x_2^0 + r x_3^0 = \psi(x_1^0 - x_3^0) - \varphi(x_3^0)$, there exists a unique global solution of the IVP (14), $x(t_0) = x^0$ $(x(t_0) = (I_L(t_0), U_C(t_0), I(t_0))^T)$ for the functions of the form (15), if $j \le k$, $j \le s$ and α_3 is sufficiently small, and for the functions of the form (16), if $\alpha_2 + \alpha_3 < r$.

If, additionally, $\sup\limits_{t \in [0,\infty)} |e(t)| < +\infty$ or $\int\limits_{t_0}^{+\infty} |e(t)|\,dt < +\infty$, then for the initial points (t_0, x^0) the DAE (14) is Lagrange stable (in both cases), i.e., every solution of the DAE is bounded. In particular, these requirements are fulfilled for voltages of the form

$$e(t) = \beta (t+\alpha)^{-n}, e(t) = \beta e^{-\alpha t}, e(t) = \beta e^{-\frac{(t-\alpha)^2}{\sigma^2}}, e(t) = \beta \sin(\omega t + \theta), \quad (17)$$

where $\alpha > 0$, $\beta, \sigma, \omega \in \mathbb{R}$, $n \in \mathbb{N}$, $\theta \in [0, 2\pi]$.

[M.S. Filipkovska, Lagrange stability of semilinear differential-algebraic equations and application to nonlinear electrical circuits, *Journal of Mathematical Physics, Analysis, Geometry*, 2018]

Lagrange stability. The numerical solution

$L = 500 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$, $t_0 = 0$, $x_0 = (10, -10, 5)^T$
$\varphi_0(y) = y^3$, $\varphi(y) = \sin y$, $\psi(y) = \sin y$, $h(y) = \sin y$, $e(t) = (2t + 10)^{-2}$

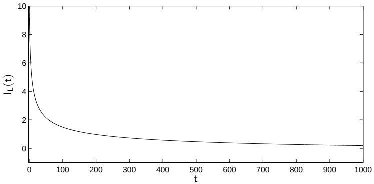

Fig. 2. The current $I_L(t)$

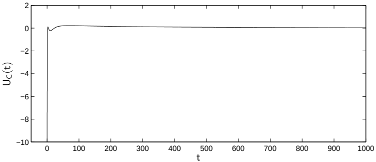

Fig. 3. The voltage $U_C(t)$

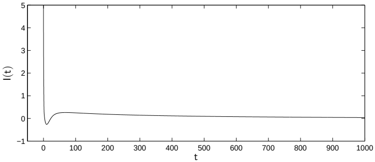

Fig. 4. The current $I(t)$

Lagrange stability. The numerical solution

$L = 500 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$, $t_0 = 0$, $x_0 = (0,0,0)^T$,
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $h(y) = y^3$, $\psi(y) = y^3$, $e(t) = 100\, e^{-t} \sin(5t)$

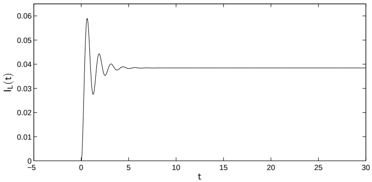

Fig. 5. The current $I_L(t)$

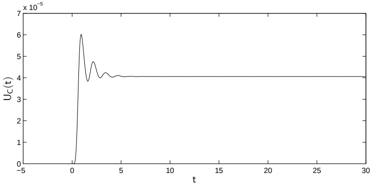

Fig. 6. The voltage $U_C(t)$

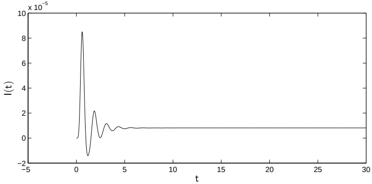

Fig. 7. The current $I(t)$

Lagrange stability. The numerical solution

$L = 300 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2.6$, $g = 0.2$, $t_0 = 0$, $x_0 = (\pi/6, 0.5, 0)^T$,
$\varphi_0(y) = y^3$, $\varphi(y) = \sin y$, $\psi(y) = \sin y$, $h(y) = \sin y$, $e(t) = 200 \sin(0.5t) - 0.2$

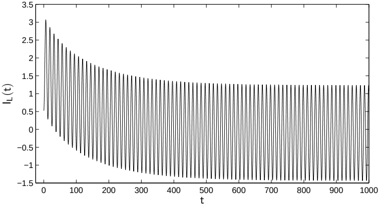

Fig. 8. The current $I_L(t)$

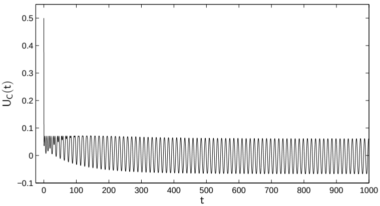

Fig. 9. The voltage $U_C(t)$

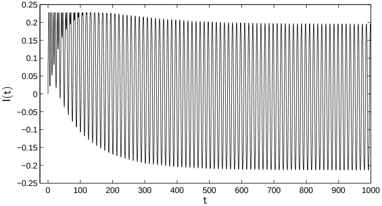

Fig. 10. The current $I(t)$

The global solution. The numerical solution

$L = 1000 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.3$, $t_0 = 0$, $x^0 = (0,0,0)^T$
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $\psi(y) = y^3$, $h(y) = y^3$, $e(t) = -t^2$

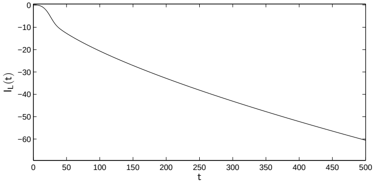

Fig. 11. The current $I_L(t)$

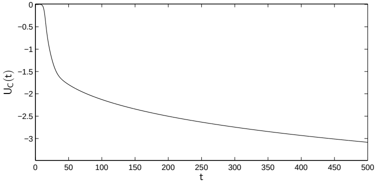

Fig. 12. The voltage $U_C(t)$

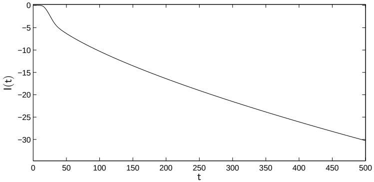

Fig. 13. The current $I(t)$

Lagrange instability. The numerical solution

$L = 10 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$,
$\varphi_0(x_1) = -x_1^2$, $\varphi(x_3) = x_3^3$, $h(x_2) = x_2^2$, $\psi(x_1 - x_3) = (x_1 - x_3)^3$, $e(t) = 2\sin t$,
$t_0 = 0$, $x_0 = (2.45, -20.625125, 2.5)^T$

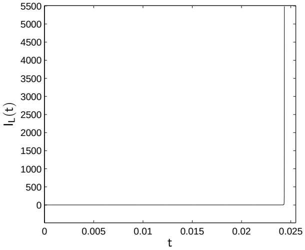

Fig. 14. The current $I_L(t)$

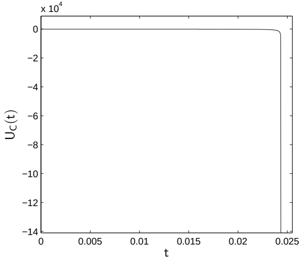

Fig. 15. The voltage $U_C(t)$

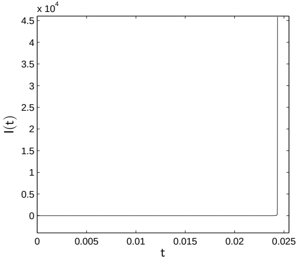

Fig. 16. The current $I(t)$

Main results:

- **Theorems on the existence and uniqueness of global solutions**

  Some advantages: the restrictions of the type of the global Lipschitz condition are not used, it is not require that the DAEs be regular DAEs of tractability index 1, and the requirements for the smoothness of the nonlinear part of the DAEs are weakened in comparison with most other similar theorems.
- **Theorem on the Lagrange stability of the DAE** (the boundedness of solutions)
- **Theorem on the Lagrange instability of the DAE** (solutions have finite escape time)
- **Theorem on the ultimate boundedness (dissipativity) of the DAE** (the ultimate boundedness of solutions)
- **Theorems on the Lyapunov stability and instability of the equilibrium state of the DAE**
- **Theorems on asymptotic stability and asymptotic stability in the large of the equilibrium state** (complete stability of the DAE)

The application of the obtained theorems to the study of certain mathematical models of electrical circuits with nonlinear and time-varying elements are shown.

Theorem 1 (the global solvability). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, and the following conditions be satisfied:

1) for each $t \in [t_+,\infty)$ and each $x_{p_1}(t) \in X_1(t)$ there exists a unique $x_{p_2}(t) \in X_2(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$;

2) for each $t_* \in [t_+,\infty)$, $x^*_{p_1}(t_*) \in X_1(t_*)$, $x^*_{p_2}(t_*) \in X_2(t_*)$ such that $(t_*, x^*_{p_1}(t_*) + x^*_{p_2}(t_*)) \in L_{t_+}$ the operator $\Phi_{t_*, x^*_{p_1}(t_*), x^*_{p_2}(t_*)} \colon X_2(t_*) \to Y_2(t_*)$, $\Phi_{t_*, x^*_{p_1}(t_*), x^*_{p_2}(t_*)} = \left[\frac{\partial}{\partial x}\left[Q_2(t_*) f(t_*, x^*_{p_1}(t_*) + x^*_{p_2}(t_*))\right] - B(t_*)\right] P_2(t_*)$, is invertible;

3) there exist a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty) \times U^c_R(0), \mathbb{R})$, where $U^c_R(0) = \{z \in \mathbb{R}^n \mid \|z\| \geq R\}$, and a function $\chi \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:

3.1) $V(t,z) \to \infty$ uniformly in t on every finite interval $[a,b) \subset [t_+,\infty)$ as $\|z\| \to \infty$,

3.2) for all t, $x_{p_1}(t)$, $x_{p_2}(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \geq R$, the inequality $V'_{(8)}(t, x_{p_1}(t)) \leq \chi\big(t, V(t, x_{p_1}(t))\big)$ holds,

3.3) the inequality $v' \leq \chi(t,v)$, $t \geq t_+$, has no positive solutions $v(t)$ with finite escape time.

Then *for each initial point* $(t_0, x_0) \in L_{t_+}$ *there exists a unique global solution of the IVP* (1), (3).

Statement 1.

Theorem 1 remains valid if the conditions 1), 2) are replaced by the following: there exists a constant $0 \le \alpha < 1$ such that

$$\left\| G^{-1}(t)\,Q_2(t)\,f\big(t, x_{p_1}(t) + x_{p_2}^1(t)\big) - G^{-1}(t)\,Q_2(t)\,f\big(t, x_{p_1}(t) + x_{p_2}^2(t)\big) \right\| \le \le \alpha \left\| x_{p_2}^1(t) - x_{p_2}^2(t) \right\| \quad (18)$$

for any $t \in [t_+, \infty)$, $x_{p_1}(t) \in X_1(t)$ and $x_{p_2}^i(t) \in X_2(t)$, $i = 1,2$.

Theorem 2 (the global solvability).

Theorem 1 remains valid if the conditions 1), 2) are replaced by the following:

1) for each $t \in [t_+, \infty)$, $x_{p_1}(t) \in X_1(t)$ there exists $x_{p_2}(t) \in X_2(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$;

2) for each $t_* \in [t_+, \infty)$, $x_{p_1}^*(t_*) \in X_1(t_*)$, $x_{p_2}^i(t_*) \in X_2(t_*)$ such that $(t_*, x_{p_1}^*(t_*) + x_{p_2}^i(t_*)) \in L_{t_+}$, $i = 1,2$, the operator function $\Phi_{t_*, x_{p_1}^*(t_*)}(x_{p_2}(t_*))$ defined by

$$\Phi_{t_*, x_{p_1}^*(t_*)} : X_2(t_*) \to L(X_2(t_*), Y_2(t_*)),$$

$$\Phi_{t_*, x_{p_1}^*(t_*)}(x_{p_2}(t_*)) = \left[\frac{\partial}{\partial x} \left[Q_2(t_*) f(t_*, x_{p_1}^*(t_*) + x_{p_2}(t_*)) \right] - B(t_*) \right] P_2(t_*), \quad (19)$$

is basis invertible on $[x_{p_2}^1(t_*), x_{p_2}^2(t_*)]$.

A system of one-dimensional projectors $\{\Theta_k\}_{k=1}^s$, $\Theta_k \colon Z \to Z$, such that $\Theta_i \Theta_j = \delta_{ij} \Theta_i$ (δ_{ij} is the Kronecker delta), and $E_Z = \sum_{k=1}^{s} \Theta_k$ is called an *additive resolution of the identity* in s-dimensional linear space Z.

Let W, Z be s-dimensional linear spaces, $D \subset W$ and $\hat{w}, \hat{\hat{w}} \in D$.

An operator function $\Phi \colon D \to L(W,Z)$ is called **basis invertible** on the interval $[\hat{w}, \hat{\hat{w}}]$, if for any set $\{w^k\}_{k=1}^s$, $w^k \in [\hat{w}, \hat{\hat{w}}]$, and some additive resolution of the identity $\{\Theta_k\}_{k=1}^s$ in the space Z the operator $\Lambda = \sum_{k=1}^{s} \Theta_k \Phi(w^k) \in L(W,Z)$ has an inverse $\Lambda^{-1} \in L(Z,W)$.

If we represent $\Phi(w) \in L(W,Z)$ as a matrix relative to some bases in W, Z:

$$\Phi(w) = \begin{pmatrix} \Phi_{11}(w) & \cdots & \Phi_{1s}(w) \\ \cdots & \cdots & \cdots \\ \Phi_{s1}(w) & \cdots & \Phi_{ss}(w) \end{pmatrix},$$ then the operator Λ takes the form

$$\Lambda = \begin{pmatrix} \Phi_{11}(w^1) & \cdots & \Phi_{1s}(w^1) \\ \cdots & \cdots & \cdots \\ \Phi_{s1}(w^s) & \cdots & \Phi_{ss}(w^s) \end{pmatrix}.$$

[A.G. Rutkas, M.S. Filipkovska, Extension of solutions of one class of differential-algebraic equations, *Journal of Computational and Applied Mathematics*, 2013] (Russian)

Theorem 3 (Lagrange stability). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, the requirements 1), 2) of Theorem 1 or 2 be fulfilled, and

3) there exists a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty) \times U_R^c(0), \mathbb{R})$ and a function $\chi \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:

3.1) $V(t,z) \to \infty$ uniformly in t on $[t_+,\infty)$ as $\|z\| \to \infty$;

3.2) for all t, $x_{p_1}(t)$, $x_{p_2}(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \geq R$, the inequality $V'_{(8)}(t, x_{p_1}(t)) \leq \chi\big(t, V(t, x_{p_1}(t))\big)$ holds;

3.3) the differential inequality $v' \leq \chi(t,v)$, $t \geq t_+$, has no unbounded positive solutions $v(t)$ for $t \in [t_+,\infty)$.

Let one of the following conditions be also satisfied:

4.a) for all $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \leq M < \infty$ (M is an arbitrary constant), the inequality $\|G^{-1}(t) Q_2(t) [f(t, x_{p_1}(t) + x_{p_2}(t)) - A'(t) x_{p_1}(t)]\| \leq K_M < \infty$, where $K_M = K(M)$ is some constant, holds;

4.b) for all $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \leq M < \infty$, the inequality $\|x_{p_2}(t)\| \leq K_M < \infty$, where $K_M = K(M)$ is some constant, holds.

Then **the equation (1) is Lagrange stable**.

Theorem 4 (Lagrange instability). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, the requirements 1), 2) of Theorem 1 or 2 be fulfilled, and

3) there exists a region $\Omega \subset \mathbb{R}^n$, $0 \notin \Omega$, such that the component $P_1(t)x(t)$ of each existing solution $x(t)$ with the initial point $(t_0,x_0) \in L_{t_+}$, where $P_1(t_0)x_0 \in \Omega$, remains all the time in Ω;

4) there exist a positive definite function $V \in C^1([t_+,\infty) \times \Omega, \mathbb{R})$ and a function $\chi \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:

4.1) for all t, $x_{p_1}(t)$, $x_{p_2}(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $x_{p_1}(t) \in \Omega$, the inequality $V'_{(8)}(t, x_{p_1}(t)) \geq \chi\big(t, V(t, x_{p_1}(t))\big)$ holds,

4.2) the inequality $v' \geq \chi(t,v)$, $t \geq t_+$, has no positive solutions defined in the future (i.e., defined for all $t \geq t_+$).

Then **for each initial point** $(t_0,x_0) \in L_{t_+}$ **such that** $P_1(t_0)x_0 \in \Omega$**, there exists a unique solution of the IVP** (1), (3) **and this solution is Lagrange unstable**.

Remarks on the form of the functions χ

It is usually convenient to choose $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ in the form

$$\chi(t,v) = k(t)U(v), \tag{20}$$

where $U \in C(0,\infty)$, $k \in C([t_+,\infty),\mathbb{R})$. Then the theorem conditions can be changed as follows:

- in Theorems 1, 2 on the global solvability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} = \infty$ ($c > 0$ is some constant) instead of the condition 3.3);
- in Theorem 3 on the Lagrange stability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} = \infty$ and $\int\limits_{t_0}^\infty k(t)dt < \infty$ ($t_0 \ge t_+$ is some number) instead of the condition 3.3);
- in Theorem 4 on the Lagrange instability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} < \infty$ and $\int\limits_{t_0}^\infty k(t)dt = \infty$ instead of the condition 4.2).

[Filipkovskaya M. S. Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. I, *Differential Equations*, 2021]
[Filipkovskaya M. S. Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. II, *Differential Equations*, 2021]

Theorem 5 (uniform dissipativity (ultimate boundedness)). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, the requirements 1), 2) of Theorem 1 or 2 be fulfilled, and

3) there exist a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty) \times U_R^c(0), \mathbb{R})$ and functions $U_j \in C([0,\infty))$, $j = 0,1,2$, such that $U_0(r)$ is non-decreasing and $U_0(r) \to +\infty$ as $r \to +\infty$, $U_1(r)$ is increasing, $U_2(r) > 0$ for $r > 0$, and for all $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$, $x_{p_2}(t) \in X_2(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \geq R$ the condition $U_0(\|x_{p_1}(t)\|) \leq V(t, x_{p_1}(t)) \leq U_1(\|x_{p_1}(t)\|)$ and one of the following inequalities hold:

3.a) $V'_{(8)}(t, x_{p_1}(t)) \leq -U_2\big(\|x_{p_1}(t)\|\big)$;

3.b) $V'_{(8)}(t, x_{p_1}(t)) \leq -U_2\big((H(t)x_{p_1}(t), x_{p_1}(t))\big)$, where $H \in C([t_+,\infty), L(\mathbb{R}^n))$ is some self-adjoint positive definite operator function such that $\sup\limits_{t \in [t_+,\infty)} \|H(t)\| < \infty$;

3.c) $V'_{(8)}(t, x_{p_1}(t)) \leq -C\,V(t, x_{p_1}(t))$, where $C > 0$ is some constant;

4) there exist a constant $c > 0$ and a number $T > t_+$ such that $\|G^{-1}(t)Q_2(t)[f(t, x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)]\| \leq c\,\|x_{p_1}(t)\|$ for all $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_T$.

Then **the DAE** (1) **is uniformly ultimately bounded (uniformly dissipative)**.

Remarks on the form of the functions V

It is often convenient to choose the positive definite scalar function $V(t,z)$ in the form

$$V(t,z) = (H(t)z,z), \tag{21}$$

where $H \in C^1([t_+,\infty),L(\mathbb{R}^n))$ is a self-adjoint positive definite operator function. The function $V(t,z)$ (21) satisfies the conditions (except for the conditions on the derivative of the function along the trajectories of (8)) of Theorems 1–4 on the global solvability, the Lagrange stability and the Lagrange instability, and if additionally $\sup\limits_{t\in[t_+,\infty)} \|H(t)\| < \infty$, then the function (21) also satisfies the conditions of Theorem 5 on the dissipativity.

[Filipkovska (Filipkovskaya) M. S. Global boundedness and stability of solutions of nonautonomous degenerate differential equations, *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan*, 2020]

It is known that the dynamics of electrical circuits is modeled using systems of differential and algebraic equations, which in a vector form have the form of differential-algebraic equations. Generally, a DAE describing the dynamics of an electrical circuit cannot be reduced to a purely differential equation, i.e., to an explicit ODE.

A simple electrical circuit with nonlinear and time-varying elements

Consider the electrical circuit with a time-varying inductance $\mathrm{L}(\mathrm{t})$, time-varying linear resistances $\mathrm{r}(\mathrm{t})$, $\mathrm{r_L}(\mathrm{t})$ and nonlinear resistances $\varphi_{\mathrm{L}}(\mathrm{I_L})$, $\varphi(\mathrm{I}_\varphi)$, whose dynamics is described by the system of equations

$$\frac{\mathrm{d}}{\mathrm{dt}}[\mathrm{L}(\mathrm{t})\mathrm{x}_1(\mathrm{t})] + \mathrm{r_L}(\mathrm{t})\mathrm{x}_1(\mathrm{t}) - \mathrm{x}_2(\mathrm{t}) = -\varphi_{\mathrm{L}}(\mathrm{x}_1(\mathrm{t})), \quad (22)$$

$$\mathrm{x}_1(\mathrm{t}) + \mathrm{x}_3(\mathrm{t}) = \mathrm{I}(\mathrm{t}), \quad (23)$$

$$\mathrm{x}_2(\mathrm{t}) - \mathrm{r}(\mathrm{t})\mathrm{x}_3(\mathrm{t}) = \mathrm{U}(\mathrm{t}) + \varphi(\mathrm{x}_3(\mathrm{t})), \quad (24)$$

where $\mathrm{I}(\mathrm{t})$ is a given (input) current, $\mathrm{U}(\mathrm{t})$ is a given (input) voltage, $\mathrm{x}_1(\mathrm{t}) = \mathrm{I_L}(\mathrm{t})$ and $\mathrm{x}_3(\mathrm{t}) = \mathrm{I}_\varphi(\mathrm{t})$ are unknown currents, and $\mathrm{x}_2(\mathrm{t}) = \mathrm{U_L}(\mathrm{t})$ is an unknown voltage. The remaining currents and voltages in the circuit are uniquely expressed in terms of $\mathrm{I}(\mathrm{t})$, $\mathrm{I_L}(\mathrm{t})$, $\mathrm{I}_\varphi(\mathrm{t})$, $\mathrm{U}(\mathrm{t})$ and $\mathrm{U_L}(\mathrm{t})$.

The vector form of the system (22)–(24) is the time-varying semilinear DAE (1):
$$\frac{d}{dt}[A(t)x]+B(t)x=f(t,x),$$
where $x=(x_1,x_2,x_3)^T=(I_L,U_L,I_\varphi)^T\in\mathbb{R}^3$,

$$A(t)=\begin{pmatrix}L(t)&0&0\\0&0&0\\0&0&0\end{pmatrix},B(t)=\begin{pmatrix}r_L(t)&-1&0\\1&0&1\\0&1&-r(t)\end{pmatrix},f(t,x)=\begin{pmatrix}-\varphi_L(x_1)\\I(t)\\U(t)+\varphi(x_3)\end{pmatrix}. \quad (25)$$

The initial condition (3): $x(t_0)=x_0,\quad x_0=(I_L(t_0),U_L(t_0),I_\varphi(t_0))^T$.

The projection matrices $P_i(t)$, $Q_i(t)$ have the form $P_1(t)=\begin{pmatrix}1&0&0\\-r(t)&0&0\\-1&0&0\end{pmatrix}$,

$$P_2(t)=\begin{pmatrix}0&0&0\\r(t)&1&0\\1&0&1\end{pmatrix},\ Q_1(t)=\begin{pmatrix}1&r(t)&1\\0&0&0\\0&0&0\end{pmatrix},\ Q_2(t)=\begin{pmatrix}0&-r(t)&-1\\0&1&0\\0&0&1\end{pmatrix}.$$

The vector x has the projections $x_{p_1}(t)=P_1(t)x=(x_1,-r(t)x_1,-x_1)^T$, $x_{p_2}(t)=P_2(t)x=(0,r(t)x_1+x_2,x_1+x_3)^T$.

Denote $a=x_1$, $b(t)=r(t)x_1+x_2$, $c=x_1+x_3$, then $x_{p_1}(t)=a(1,-r(t),-1)^T$, $x_{p_2}(t)=(0,b(t),c)^T$.

By **Theorem 1** as well as by **Theorem 2**, for each initial point $(t_0,x_0) \in [t_+,\infty) \times \mathbb{R}^3$ satisfying the algebraic equations (23), (24) (i.e., $(t_0,x_0) \in L_{t_+}$), *there exists a unique global solution* $x(t)$ of the equation (1) satisfying the IVP (1), (3) if $L, r, r_L \in C^1([t_+,\infty),\mathbb{R})$, $I, U \in C([t_+,\infty),\mathbb{R})$, $\varphi, \varphi_L \in C^1(\mathbb{R})$ and the following requirements are fulfilled:

$L(t) \geq L_0 > 0$ and $r(t) \neq 0$ for all $t \in [t_+,\infty)$;

$\lambda L(t) + r_L(t) + r(t) \neq 0$ for sufficiently large $|\lambda|$ such that $|\lambda| \geq L_0^{-1}$ and all $t \in [t_+,\infty)$;

there exists a number $R > 0$ such that the inequality

$$[\varphi_L(x_1) - \varphi(I(t) - x_1) - r(t)I(t) - U(t)]x_1 + [L'(t)/2 + r_L(t) + r(t)]x_1^2 \geq 0$$

holds for all $t \in [t_+,\infty)$, $\|x_{p_1}(t)\| = |x_1|\|(1,-r(t),-1)^T\| \geq R$.

If, additionally, $\int\limits_{t_0}^{\infty} k(t)dt < \infty$, where $k(t) = |r'(t)/r(t)|$, and the functions $I_2(t)$, $U_1(t)$, $r(t)$ are bounded for all $t \in [t_+,\infty)$, i.e., $\sup\limits_{t\in[t_+,\infty)} |I_2(t)| < \infty$, $\sup\limits_{t\in[t_+,\infty)} |U_1(t)| < \infty$, $\sup\limits_{t\in[t_+,\infty)} |r(t)| < \infty$, then *the DAE* (1) (with $A(t)$, $B(t)$, $f(t,x)$ of the form (25)) *is Lagrange stable* by **Theorem 3**.

The mathematical model of a time-varying nonlinear electrical circuit

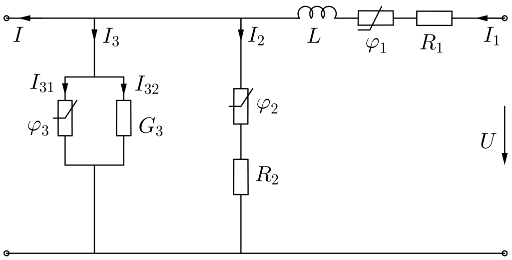

Fig. 17. The diagram of the electric circuit

A current $\mathrm{I}(\mathrm{t})$, a voltage $\mathrm{U}(\mathrm{t})$, resistances $\mathrm{R}_1(\mathrm{t})$, $\mathrm{R}_2(\mathrm{t})$, $\varphi_1(\mathrm{I}_1)$, $\varphi_2(\mathrm{I}_2)$, $\varphi_3(\mathrm{I}_{31})$, a conductance $\mathrm{G}_3(\mathrm{t})$, an inductance $\mathrm{L}(\mathrm{t})$ and a capacitance C are given.

A transient process in the electrical circuit (Fig. 17) is described by the system

$$\frac{\mathrm{d}}{\mathrm{dt}}[\mathrm{L}(\mathrm{t})\mathrm{I}_1(\mathrm{t})]+\mathrm{R}_1(\mathrm{t})\mathrm{I}_1(\mathrm{t})=\mathrm{U}(\mathrm{t})-\varphi_1(\mathrm{I}_1(\mathrm{t}))-\varphi_3(\mathrm{I}_{31}(\mathrm{t})), \tag{26}$$

$$\mathrm{I}_1(\mathrm{t})-\mathrm{I}_{31}(\mathrm{t})-\mathrm{I}_2(\mathrm{t})=\mathrm{I}(\mathrm{t})+\mathrm{G}_3(\mathrm{t})\varphi_3(\mathrm{I}_{31}(\mathrm{t})), \tag{27}$$

$$\mathrm{R}_2(\mathrm{t})\mathrm{I}_2(\mathrm{t})=\varphi_3(\mathrm{I}_{31}(\mathrm{t}))-\varphi_2(\mathrm{I}_2(\mathrm{t})), \tag{28}$$

Denote $x_1(t) = I_1(t)$, $x_2(t) = I_{31}(t)$ and $x_3(t) = I_2(t)$.

The vector form of the system (26)–(28) is the time-varying semilinear DAE (1):

$$\frac{d}{dt}[A(t)x] + B(t)x = f(t,x),$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A(t) = \begin{pmatrix} L(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B(t) = \begin{pmatrix} R_1(t) & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & R_2(t) \end{pmatrix},$$

$$f(t,x) = \begin{pmatrix} U(t) - \varphi_1(x_1) - \varphi_3(x_2) \\ I(t) + G_3(t)\varphi_3(x_2) \\ \varphi_3(x_2) - \varphi_2(x_3) \end{pmatrix}.$$

The initial condition (3): $x(t_0) = x_0$, $x_0 = (I_1(t_0), I_{31}(t_0), I_2(t_0))^T$.

It is assumed that the functions $L(t)$, $R_1(t)$, $R_2(t)$ and $G_3(t)$ are positive for all $t \in [t_+, \infty)$.

The projections $x_{p_j}(t) = P_j(t)x \in X_j(t)$ of a vector x have the form $x_{p_1}(t) = x_{p_1} = (x_1, x_1, 0)^T$, $x_{p_2}(t) = x_{p_2} = (0, x_2 - x_1, x_3)^T$.

Denote $z = x_1$, $u = x_2 - x_1$, $w = x_3$, then $x_{p_1} = (z,z,0)^T$, $x_{p_2} = (0,u,w)^T$.

Using the introduced notation, the equations (27)–(28) can be rewritten as

$$w = -I(t) - u - G_3(t)\,\varphi_3(u+z), \tag{29}$$

$$u = \psi(t,z,u), \quad \text{where} \quad \psi(t,z,u) = -I(t) - \big(G_3(t) + R_2^{-1}(t)\big)\,\varphi_3(u+z) + $$
$$+ R_2^{-1}(t)\,\varphi_2\big(-I(t) - u - G_3(t)\varphi_3(u+z)\big). \tag{30}$$

By **Theorem 1** for each initial point $(t_0,x_0) \in [t_+,\infty) \times \mathbb{R}^3$ satisfying the algebraic equations (i.e., $(t_0,x_0) \in L_{t_+}$)

$$x_1 - x_2 - x_3 = I(t) + G_3(t)\,\varphi_3(x_2), \tag{31}$$
$$R_2(t)\,x_3 = \varphi_3(x_2) - \varphi_2(x_3), \tag{32}$$

there exists a unique global solution $x(t)$ of the IVP (1), (3) if
$L, R_1, R_2 \in C^1([t_+,\infty),\mathbb{R})$, $I, U, G_3 \in C([t_+,\infty),\mathbb{R})$, $\varphi_j \in C^1(\mathbb{R})$, $j = 1,2,3$;
$L(t) > 0$, $R_1(t) > 0$, $R_2(t) > 0$, $G_3(t) > 0$ for all $t \in [t_+,\infty)$;

1) for each $t \in [t_+,\infty)$ and each $z \in \mathbb{R}$ there exists a unique $u \in \mathbb{R}$ satisfying the equality (30);

2) for each $t_* \in [t_+,\infty)$, $z_* \in \mathbb{R}$ and each $u_*, w_* \in \mathbb{R}$ satisfying the equalities (29), (30), one has the relation

$$\varphi_3'(u_* + z_*) + \big[\varphi_2'(w_*) + R_2(t_*)\big]\big[1 + G_3(t_*)\,\varphi_3'(u_* + z_*)\big] \neq 0; \tag{33}$$

3) there exists $R > 0$ such that $-\big(\varphi_1(z) + \varphi_3(u+z)\big)z \le R_1(t)z^2$ for all $t \in [t_+,\infty)$, $u, w \in \mathbb{R}$, $z \in \mathbb{R}$, $|z| \ge R$, satisfying the equalities (29), (30).

A similar assertion takes place according to **Theorem 2**, if the above conditions are satisfied with the following changes: the condition 1) does not contain the requirement that u be unique; the condition 2) is replaced by the following: 2*) for each $t_* \in [t_+,\infty)$, $z_* \in \mathbb{R}$ and each $u_*^j, w_*^j \in \mathbb{R}$, $j = 1,2$, satisfying the equalities (29), (30), the relation

$$\varphi_3'(u_2 + z_*) + \left[\varphi_2'(w_2) + R_2(t_*)\right]\left[1 + G_3(t_*)\,\varphi_3'(u_1 + z_*)\right] \neq 0$$

holds for any $u_k \in [u_*^1, u_*^2]$, $w_k \in [w_*^1, w_*^2]$, $k = 1,2$.

If, additionally, $\int\limits_{t_0}^{\infty} k(t)dt < \infty$, where $k(t) = 2L^{-1}(t)\,(|L'(t| + |U(t)|)$, the functions $I(t)$, $R_2^{-1}(t)$, $G_3(t)$ are bounded for all $t \in [t_+,\infty)$, and $\varphi_3(x_2)$, $\varphi_2(x_3)$ are bounded for $x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}$ respectively, then *the DAE* (1) *is Lagrange stable* by **Theorem 3**.

The particular cases.

The the conditions 1), 2) and 2*) are satisfied for the functions φ_2, φ_3 which are increasing (nondecreasing) on $\mathbb{R}$, for example,

$$\varphi_2(y) = a\,y^{2k-1},\, \varphi_3(y) = b\,y^{2m-1},\, \varphi_1(y) = c\,y^{2l-1}, \quad a,b,c > 0,\, k,m,l \in \mathbb{N}, \quad (34)$$

and if b is sufficiently small, $m \le l$, $\sup\limits_{t\in[t_+,\infty)} |I(t)| < \infty$ and $R_2(t) \ge K_0 = \mathrm{const} > 0$, $t \in [t_+,\infty)$, then the condition 3) is also fulfilled.

Note that in this case the mapping $\psi(t,z,u)$ is not globally contractive with respect to u. Obviously, the condition 1) is satisfied, if $\psi(t,z,u)$ is globally contractive with respect to u for any t, z, i.e., there exists a constant $\alpha < 1$ such that $\left|\psi(t,z,u_1) - \psi(t,z,u_2)\right| \le \alpha|u_1 - u_2|$ for any $t \in [t_+,\infty)$, $z \in \mathbb{R}$, $u_1, u_2 \in R$.

References I

1. Filipkovskaya M. S. *Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. I*, Differential Equations. – 2021. – Vol. 57, No. 1. – P. 19-40.
2. Filipkovskaya M. S. *Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. II*, Differential Equations. – 2021. – Vol. 57, No. 2. – P. 196–209.
3. Filipkovska (Filipkovskaya) M. S. *Global boundedness and stability of solutions of nonautonomous degenerate differential equations*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan. – 2020. – Vol. 46, No. 2. – P. 243–271.
4. Filipkovska M.S., *Lagrange stability of semilinear differential-algebraic equations and application to nonlinear electrical circuits*, Journal of Mathematical Physics, Analysis, Geometry, 2018, Vol. 14, No 2, 1–28.

References II

5. Filipkovska M.S. Two combined methods for the global solution of implicit semilinear differential equations with the use of spectral projectors and Taylor expansions, Int. J. of Computing Science and Mathematics. DOI: 10.1504/IJCSM.2019.10025236 [in press].
6. Rutkas A.G., Filipkovska M.S., Extension of solutions of one class of differential-algebraic equations, *Journal of Computational and Applied Mathematics*, 2013, No. 1(111), 135–145. (Russian)
7. La Salle J., Lefschetz S., *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.
8. T. Yoshizawa, *Stability theory by Liapunov's second method*, Math. Soc. Jpn., Tokyo, 1966.
9. Rutkas A.G., Vlasenko L.A. *Existence of solutions of degenerate nonlinear differential operator equations*, Nonlinear Oscillations. 2001. Vol. 4, No. 2, P. 252–263.

Thank you for your attention!