On decay rates of solutions of parabolic Cauchy problems

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Papers

This presentation is based on results in the following of papers published together with José Bonet (Valencia) and Wolfgang Lusky (Paderborn).

[1] J.Bonet, W.Lusky, J.Taskinen, Schauder basis and the decay rate of the heat equation. J. Evol. Equations 19 (2019), 717–728.
[2] J.Bonet, W.Lusky, J.Taskinen: On decay rates of the solutions of parabolic Cauchy problems, Proc. Royal Soc. Edinburgh, to appear. DOI:10.1017/prm.2020.48

Abstract

• We consider the Cauchy problem in the Euclidean space $\mathbb{R}^N \ni x$ for the parabolic equation $\partial_t u(x,t) = Au(x,t)$, where the operator A (e.g. the Laplacian) is assumed, among other things, to be a generator of a C_0 semigroup in a weighted L^p-space $L^p_w(\mathbb{R}^N)$ with $1 \le p < \infty$ and a fast growing weight w.

• We show: there is a Schauder basis $(e_n)_{n=1}^\infty$ in $L^p_w(\mathbb{R}^N)$ with the following property: given an arbitrary positive integer m there exists $n_m > 0$ such that, if the initial data f belongs to the closed linear span of e_n with $n \ge n_m$, then the decay rate of the solution of the problem is at least t^{-m} for large times t. — In other words, the Banach space of the initial data can be split into two components, where the data in the infinite-dimensional component leads to decay with any pre-determined speed t^{-m}, and the exceptional component is finite dimensional.

• Three different proofs I–III with somewhat different assumptions on A.

Main Problem under consideration

Given an integrable function $f \in L^1(\mathbb{R}^N)$ in the Euclidean space $\mathbb{R}^N$, $N \in \mathbb{N} = \{1, 2, \ldots\}$, we study the following parabolic Cauchy problem ("Main Problem") for an unknown function u on $\mathbb{R}^N \times [0, \infty) \ni (x, t)$,

$$\begin{aligned} \partial_t u(x,t) &= Au(x,t) \ \text{ for } x \in \mathbb{R}^N,\ t > 0 \\ u(x,0) &= f(x) \ \text{ for } x \in \mathbb{R}^N, \end{aligned}$$

where for example $-A$ can be a strongly elliptic partial differential operator of nth order with even $n \in \mathbb{N}$: $-Ag(x) = \sum_{|\alpha| \le n} a_\alpha(x) D^\alpha g(x)$, where $a_\alpha \in L^\infty(\mathbb{R}^N)$.
We assume that A is a generator of a C_0-semigroup e^{At} in $L^1(\mathbb{R}^N)$, with an integral kernel $K : \mathbb{R}^N \times \mathbb{R}^N \times [0, \infty) \to \mathbb{C}$,

$$e^{At} f(x) = \int_{\mathbb{R}^N} K(x, y, t) f(y) dy \ , \quad x \in \mathbb{R}^N, \tag{1}$$

and that the Main Problem has a unique classical solution which coincides with (1). More assumptions on K will be posed soon.

Some more notions.

Given $M \in \mathbb{N}$, we denote by $\mathcal{B}^M(\mathbb{R}^N)$ the space of functions $h : \mathbb{R}^N \to \mathbb{R}$ such that all partial derivatives up to order M exist and are continuous and bounded. Same for $M = \infty$.

Let X denote a Banach space over the scalar field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). If X is separable, we recall that a sequence $(e_n)_{n=1}^{\infty} \subset X$ is a Schauder basis (briefly: basis), if every element $f \in X$ can be presented as a convergent sum $f = \sum_{n=1}^{\infty} f_n e_n$ where the numbers $f_n \in \mathbb{K}$ are unique for f. An orthonormal basis of a separable Hilbert space is an example.

Recall: if $N = 1$, $A = \partial_x^2$ and $f \in L^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f(y)dy = 0$, then we have

$$\|e^{t\partial_x^2} f\|_\infty \leq \frac{C}{t} \text{ instead of the typical } \|e^{t\partial_x^2} g\|_\infty \leq \frac{C}{\sqrt{t}},\ g \in L^1(\mathbb{R}).$$

I. Approach via Taylor expansion of the kernel K.

We fix $M \in \mathbb{N}$ or $M = \infty$, and assume that the semigroup generated by the operator A has an M times continuously differentiable kernel of the form

$$K(x,y,t) = \frac{d}{t^b} h\left((x-y)t^{-a}\right)$$

for some $h \in \mathcal{B}^M(\mathbb{R}^N)$, constants $a, d > 0$, $b \geq 0$, $x, y \in \mathbb{R}^N$, $t > 0$. Clearly, the Gaussian heat kernel of the Laplacian $A = \Delta$ corresponds to the case $a = 1/2$, $b = N/2$, $M = \infty$. More generally, we also consider kernels

$$K(x,y,t) = \sum_{j=1}^{J} U_j(x,t) v_j(y) h_j\left((x-y)t^{-a_j}\right), \qquad x \in \mathbb{R}^N, t > 0,$$

where $J \in \mathbb{N}$ and, for all j, the numbers $a_j > 0$ are constants, and v_j is a bounded and continuous function on $\mathbb{R}^N$, and $h_j \in \mathcal{B}^M(\mathbb{R}^N)$; finally, the measurable functions U_j are assumed to satisfy for some constants $C_j > 0$, $b_j \geq 0$,

$$U(\cdot,t) \in L^\infty(\mathbb{R}^N) \text{ for } t > 0, \quad |U_j(x,t)| \leq \frac{C_j}{t^{b_j}} \text{ for } x \in \mathbb{R}^N,\ t \geq 1.$$

I: Taylor expansion of K: weighted L^p-space for initial data

Let $1 \leq p < \infty$. We need to fix a parameter $L \in (0, \infty]$ such that

$$L > Mp + N(p-1), \text{if } M < \infty$$

and $L = \infty$, if $M = \infty$. Then, let $w_L : \mathbb{R}^N \to \mathbb{R}^+$ be a continuous weight function satisfying the growth condition

$$\sup_{x \in \mathbb{R}^N} \frac{1}{w_L(x)} (1 + |x|)^m < \infty \ \ \forall\, m \in \{1, \ldots, L\}$$

($\forall m \in \mathbb{N}$, if $L = \infty$).

We will use the Banach space $L^p_{w_L}(\mathbb{R}^N)$ with norm

$$\|f\|^p_{p,w_L} := \int_{\mathbb{R}^N} |f(x)|^p w_L(x) dx.$$

In particular, the initial data and the solution of the parabolic problem will be in this space for all times $t > 0$, and the Schauder basis will be constructed in this space.

I: Taylor expansion of K: result

Theorem

1°. $M = \infty$. There exists a basis $(e_n)_{n=1}^{\infty}$ of $L^p_{w_L}(\mathbb{R}^N)$ and an increasing sequence $(n_m)_{m=1}^{\infty} \subset \mathbb{N}$ as follows: given $m \in \mathbb{N}$ and initial data

$$f = \sum_{n=1}^{\infty} f_n e_n \in L^p_{w_L}(\mathbb{R}^N) \qquad \text{with } f_n = 0 \text{ for all } n = 1, \ldots, n_m,$$

the solution of the Main Problem has the fast decay property

$$\|e^{tA} f\|_\infty \le \frac{C_{m,p}}{t^m} \|f\|_{p,w_L} \quad \text{for all } t \ge 1.$$

2°. $M < \infty$. There exists a basis $(e_n)_{n=1}^{\infty}$ of $L^p_{w_L}(\mathbb{R}^N)$ and a number $n_M \in \mathbb{N}$ such that

$$\|e^{tA} f\|_\infty \le \frac{C_{M,p}}{t^a} \|f\|_{p,w_L} \quad \text{for all } t \ge 1,$$

where $a = \min\{Ma_j + b_j : j = 1, \ldots, J\}$ for all $f \in \overline{\mathrm{sp}}\{e_n : n \ge n_M\}$.

The proof uses the following abstract result of [1].

Theorem

Let X be a separable Banach space, let $x_m^* \in X^*$ for all $m \in \mathbb{N}$. Then, there exists an increasing sequence $(n_m)_{m=1}^{\infty} \subset \mathbb{N}$ and a basis $(e_n)_{n=1}^{\infty}$ of X such that

$$x_m^*(e_n) = 0 \quad \textit{for all} \quad n \geq n_m.$$

(The basis can actually be found as a small "perturbation" of any given "shrinking" basis of X (any basis, if X is reflexive).)

Consider $M = \infty$. We take $X = L^p_{w_L}(\mathbb{R}^N)$ and for $x_m^* \in X^*$ the functionals (which are bounded due to assumptions)

$$\Phi_{k_m}(f) := \int\limits_{\mathbb{R}^N} k_m(y) f(y) dy, \quad m \in \mathbb{N},\ f \in X,$$

where we define the functions $k_m(y) = k_{m(\alpha)}(y) := y^\alpha$, and $m(\alpha) \in \mathbb{N}$ is a numbering of the multi-indices α such that $m(\alpha) \leq m(\beta)$, if $\alpha \leq \beta$.

I: Taylor expansion of K: on the proof

Recall that the semigroup kernel is (in the simple case)

$$K(x,y,t) = \frac{d}{t^b} h\left((x-y)t^{-a}\right) =: \frac{d}{t^b} h_{x,t}(y)$$

where $h \in \mathcal{B}^M(\mathbb{R}^N)$, $a, d > 0$, $b \geq 0$.
Given arbitrary $m \in \mathbb{N}$ we make the Taylor expansion of the kernel

$$h_{x,t}(y) = \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha h_{x,t}(0) y^\alpha + R(x,y,t),$$

where $|R(x,y,t)| \leq Ct^{-ma}$. If $f = \sum_{n>n_m} f_n e_n \in X \in L^p_{w_L}(\mathbb{R}^N)$, then in the expression

$$e^{At} f(x) = \int_{\mathbb{R}^N} K(x,y,t) f(y) dy$$

$$= \sum_{|\alpha|<m, n \geq n_m} \frac{d}{\alpha! t^b} f_n D^\alpha h_{x,t}(0) \Phi_{k_{m(\alpha)}}(e_n) + \frac{d}{t^b} \int_{\mathbb{R}^N} R(x,y,t) f(y) dy$$

all but the last term vanish. The result follows from the estimate for R.

II. Approach using Fourier-analysis: assumptions

Another approach using the Fourier-transform yields a slightly different result. We fix an even $M \in \mathbb{N}$ such that $M - N \geq 2$, or $M = \infty$

We assume that the semigroup kernel is of convolution type, $K(x, y, t) = \widetilde{K}(x - y, t)$, and that in addition the function $x \mapsto D_x^\alpha \widetilde{K}(x, t)$ belongs to $L^2(\mathbb{R}^N)$ for every $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq M/2$ and $t > 0$, and that there holds for all $t \geq 1$ the estimate

$$\|D^\alpha \widetilde{K}(\cdot, t)\|_2 \leq \frac{C}{t^{a|\alpha|+b}}$$

for some constants $a > 0$ and $b \in \mathbb{R}$ and for all multi-indices α with $|\alpha| \leq M/2 - N/2$. Also, the map $t \mapsto \widetilde{K}(\cdot, t)$ should be continuous as a map from $(0, \infty)$ to $L^2(\mathbb{R}^N)$.

We again consider the continuous weight function $w_M : \mathbb{R}^N \to \mathbb{R}^+$ with

$$\sup_{x \in \mathbb{R}^N} \frac{1}{w_M(x)} (1 + |x|)^m < \infty \quad \forall\, m \in \{1, \ldots, M\}$$

($\forall m \in \mathbb{N}$, if $M = \infty$) and the corresponding Hilbert space $L^2_{w_M}(\mathbb{R}^N)$.

II. Approach using Fourier-analysis: the result

Theorem

Let the weight w_M and the kernel $\widetilde{K}$ be as described above.
1°. $M = \infty$. *There exists a basis $(e_n)_{n=1}^{\infty}$ of $L^2_{w_M}(\mathbb{R}^N)$ and an increasing sequence $(n_m)_{m=1}^{\infty}$ as follows: given $m \in \mathbb{N}$, then for any initial data*

$$f = \sum_{n=n_m}^{\infty} f_n e_n \in L^2_{w_M}(\mathbb{R}^N),$$

the solution of the Main Problem has the bound

$$\|e^{tA} f\|_\infty \leq \frac{C_m}{t^m} \|f\|_{2,w_M} \quad \text{for all } t \geq 1.$$

2°. $M < \infty$. *There exists a basis $(e_n)_{n=1}^{\infty}$ of $L^2_{w_M}(\mathbb{R}^N)$ and $n_M \in \mathbb{N}$ with*

$$\|e^{tA} f\|_\infty \leq \frac{C_M}{t^{\mu a + b}} \|f\|_{2,w_M} \quad \text{for all } t \geq 1,$$

for all $f \in \overline{\mathrm{sp}}\{e_n \, : \, n \geq n_M\}$. Here $\mu = [M/2 - N/2]$.

II. Approach using Fourier-analysis: on the proof

We again use the same linear functionals on the space $L^2_{w_M}(\mathbb{R}^N)$,

$$f \mapsto \int_{\mathbb{R}^N} f(y) y^\alpha dy.$$

Vanishing of these implies properties for the Fourier-transform of f, which together with the properties of the Fourier transform of the kernel allows us derive the needed estimates for the theorem.

III. Approach using repeated integration functionals.

Yet another approach allows us to relax the specific assumptions on the form of the x- and y-dependence of the semigroup kernel K, however, the proof only works in dimension $N = 1$ for this section. We assume that there exist constants $a > 0$ and $b \in \mathbb{R}$ such that, for some $M \in \mathbb{R} \cup \{\infty\}$,

$$|\partial_y^m K(x, y, t)| \leq \frac{C}{(t+1)^{am+b}}$$

for all $m \leq M$ (for all $m \in \mathbb{N}$, if $M = \infty$) and all $x, y \in \mathbb{R}$, $t > 0$.
The space of initial is $L^p_{w_Q}(\mathbb{R})$, where $1 \leq p < \infty$ and Q is fixed such that

$$Q > p(M+1) + 1,$$

if $M < \infty$, and $Q = \infty$, if $M = \infty$. The weight $w_Q : \mathbb{R}^N \to \mathbb{R}^+$ is as before.

III. Approach using repeated integration functionals.

Theorem

1°. $M = \infty$. *There exists a basis* $(e_n)_{n=1}^\infty$ *of* $L^p_{w_Q}(\mathbb{R})$ *and an increasing sequence* $(n_m)_{m=1}^\infty$ *as follows: given* $m \in \mathbb{N}$, *then for any initial data*

$$f = \sum_{n=n_m}^\infty f_n e_n \in L^p_{w_Q}(\mathbb{R}),$$

the solution of the Main Problem has the estimate

$$\|e^{tA} f\|_\infty \le \frac{C_m}{t^m} \|f\|_{p,w_Q} \quad \text{for all } t \ge 1.$$

2°. $M < \infty$. *There exists a basis* $(e_n)_{n=1}^\infty$ *of* $L^p_{w_Q}(\mathbb{R})$ *and a number* $n_M \in \mathbb{N}$ *such that*

$$\|e^{tA} f\|_\infty \le \frac{C_M}{t^{Ma+b}} \|f\|_{p,w_Q} \quad \text{for all } t \ge 1,$$

for all $f \in \mathrm{sp}\{e_n \,:\, n \ge n_M\}$.

FINALE:

Thank you for your attention!