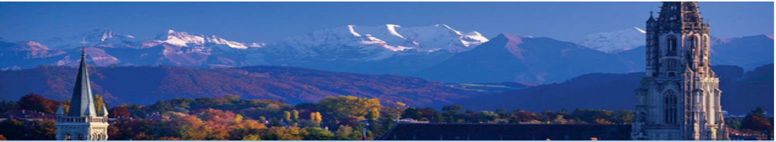

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Discontinuous Galerkin Discretisations for Problems with Dirac Delta Source

MS39 – Modeling, approximation, and analysis of partial differential equations involving singular source terms

Paul Houston and Thomas P. Wihler

8ECM – June 21, 2021

Problem formulation

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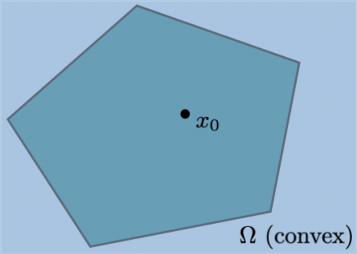
Find $u \in \mathrm{W}_0^{1,p}(\Omega)$ s.t.

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = v(x_0) \quad \forall v \in \mathrm{W}_0^{1,p^\star}(\Omega).$$

$(1 \leq p < 2)$



DG discretisation

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- DG space:

$$\mathbb{V}_{\mathsf{DG}}(\mathcal{T}) = \{v \in \mathrm{L}^2(\Omega) : v|_K \in \mathbb{S}_\ell(K) \quad \forall K \in \mathcal{T}\}$$

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$$\underbrace{\int_\Omega \nabla_{\mathcal{T}} u_{\mathsf{DG}} \cdot \nabla_{\mathcal{T}} v \, \mathsf{d}x + \mathfrak{F}_{\mathsf{DG}}(u_{\mathsf{DG}}, v)}_{=: a_{\mathsf{DG}}(u_{\mathsf{DG}}, v)} = v(x_0) \quad \forall v \in \mathbb{V}_{\mathsf{DG}}(\mathcal{T})$$

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with SIPG-fluxes

$$\mathfrak{F}_{\mathsf{DG}}(u_{\mathsf{DG}}, v) = -\int_{\mathcal{E}} \left(\langle\!\langle \nabla_{\mathcal{T}} u_{\mathsf{DG}} \rangle\!\rangle \cdot [\![v]\!] + [\![u_{\mathsf{DG}}]\!] \cdot \langle\!\langle \nabla_{\mathcal{T}} v \rangle\!\rangle \,\mathsf{d}s \right) \mathsf{d}s$$
$$+ \gamma \int_{\mathcal{E}} \mathtt{h}^{-1} [\![u_{\mathsf{DG}}]\!] \cdot [\![v]\!] \,\mathsf{d}s$$

Approximating δ_{x_0}

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$$\int_{K_0} \delta_{K_0} v \, \mathsf{d}x = v(\boldsymbol{x}_0) \qquad \forall v \in \mathbb{S}_\ell(K_0)$$

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we have

$$\int_\Omega \delta_{\mathsf{DG}} v \,\mathsf{d}x = v(\boldsymbol{x}_0) \quad \forall v \in \mathbb{V}_{\mathsf{DG}}(\mathcal{T}) \tag{CP}$$

A priori error analysis I

u^b

^{b} UNIVERSITÄT BERN

Auxiliary problem: $U^h \in \mathrm{H}^1_0(\Omega)$ s.t.

$$\int_\Omega \nabla U^h \cdot \nabla v \, \mathsf{d}x = \int_\Omega \delta_{\mathsf{DG}} v \, \mathsf{d}x \qquad \forall v \in \mathrm{H}^1_0(\Omega)$$

[1] Scott; Numer. Math.; 1973/74

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$$\|U^h\|_{\mathrm{H}^2(\Omega)} \leq C \|\delta_{\mathsf{DG}}\|_{\mathrm{L}^2(\Omega)} \sim h_{K_0}^{-1} \qquad \text{(Reg)}$$

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> Approximation property on quasi-uniform meshes [1]:

$$\|u - U^h\|_{\mathrm{L}^2(\Omega)} \leq C(\partial\Omega, x_0)h \qquad \text{(AP)}$$

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A priori error analysis II

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- DG formulation & (CP):

$$a_{\mathsf{DG}}(u_{\mathsf{DG}}, v) = v(\boldsymbol{x}_0) = \int_\Omega \delta_{\mathsf{DG}} v \,\mathsf{d}x \quad \forall v \in \mathbb{V}_{\mathsf{DG}}(\mathcal{T})$$

[2] Arnold, Brezzi, Cockburn, Marini; SIAM J. Numer. Anal.; 2001

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- Error bound [2]:

$$\|U^h - u_{\mathsf{DG}}\|_{\mathrm{L}^2(\Omega)} \leq Ch^2 \|U^h\|_{\mathrm{H}^2(\Omega)} \overset{\text{(Reg)}}{\leq} Ch^2 \underbrace{\|\delta_{\mathsf{DG}}\|_{\mathrm{L}^2(\Omega)}}_{\sim h^{-1}} \sim h$$

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- Triangle inequality:

$$\|u - u_{\mathsf{DG}}\|_{\mathrm{L}^2(\Omega)} \le \|u - U^h\|_{\mathrm{L}^2(\Omega)} + \|U^h - u_{\mathsf{DG}}\|_{\mathrm{L}^2(\Omega)} \overset{\text{(AP)}}{\le} Ch$$

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Numerical experiment

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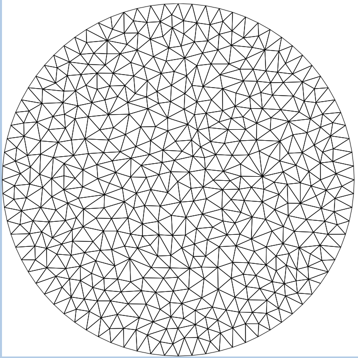
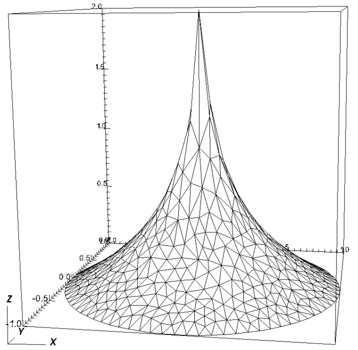
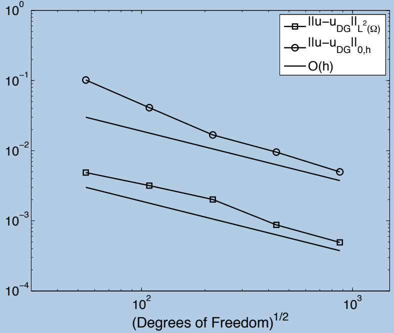
Fundamental solution of Laplacian (988 elements):

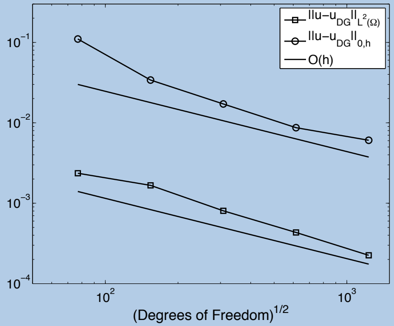




Numerical experiment—$\mathcal{O}(h)$ convergence



$\ell = 1$



$\ell = 2$

$$\|u - u_{\mathrm{DG}}\|_{0,h}^2 \sim \|u - u_{\mathrm{DG}}\|_{\mathrm{L}^2(\Omega)}^2 + \|\mathbf{h}^{1/2}[\![u - u_{\mathrm{DG}}]\!]\|_{\mathrm{L}^2(\partial\Omega)}^2$$

Mesh refinements

> Error bound may be improved on *graded meshes*

Apel, Benedix, Sirch, Vexler; SIAM J. Numer. Anal.; 2011

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> A posteriori error estimate:

$$\|u - u_{\mathsf{DG}}\|_{\mathrm{L}^2(\Omega)} \leq C\Big(h_{K_0}^2 + \sum_{K\in\mathcal{T}} \eta_K\Big)^{1/2},$$

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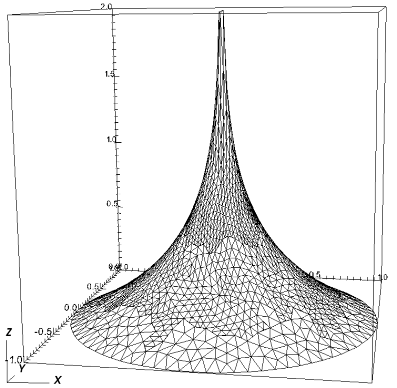
with local error indicators

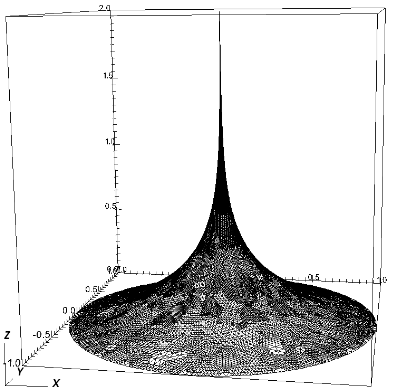
$$\eta_K := h_K^4 \|\Delta u_{\mathsf{DG}} + \delta_{\mathsf{DG}}\|_{\mathrm{L}^2(K)}^2 + h_K^3 \|[\![\nabla_{\mathcal{T}} u_{\mathsf{DG}}]\!]\|_{\mathrm{L}^2(\partial K \setminus \partial\Omega)}^2 + h_K \|[\![u_{\mathsf{DG}}]\!]\|_{\mathrm{L}^2(\partial K)}^2,$$

Houston & W.; ESAIM: M2AN; 2012

Numerical experiment



2 adaptive refinements



6 adaptive refinements

Singular boundary conditions

Poisson problem with discontinuous boundary condition:

$$-\Delta u(x,y) = 0 \qquad\qquad -1 < x < 1,\, 0 < y < 1$$

$$g(x, y=0) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$$

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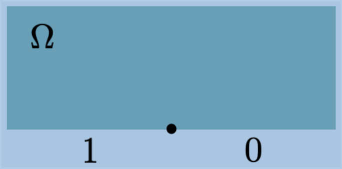
Singular behavior of exact solution at origin:

$$u(r,\theta) = \frac{1}{\pi}\theta = \frac{1}{\pi}\Im(\log(x+iy))$$

$$\nabla u(r,\theta) = \frac{1}{\pi r}\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$



Singular boundary conditions

DG discretisation:

$$a_{\mathsf{DG}}(u_{\mathsf{DG}}, v) = -\int_{\partial\Omega} (\nabla_{\mathcal{T}} v \cdot \boldsymbol{n}) g \, \mathsf{d}s + \gamma \int_{\partial\Omega} \mathtt{h}^{-1} g v \, \mathsf{d}s \quad \forall v \in \mathbb{V}_{\mathsf{DG}}(\mathcal{T})$$

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Local a posteriori error indicators:

$$\eta_K = h_K^4 \|\Delta u_{\mathsf{DG}}\|^2_{\mathrm{L}^2(K)} + h_K^3 \|[\![\nabla_{\mathcal{T}} u_{\mathsf{DG}}]\!]\|^2_{\mathrm{L}^2(\partial K \setminus \partial\Omega)} + h_K \|[\![u_{\mathsf{DG}}]\!]\|^2_{\mathrm{L}^2(\partial K \setminus \partial\Omega)} + h_K \|g - u_{\mathsf{DG}}\|^2_{\mathrm{L}^2(\partial K \cap \partial\Omega)}$$

Houston & W.; IMA J. Numer. Anal.; 2011

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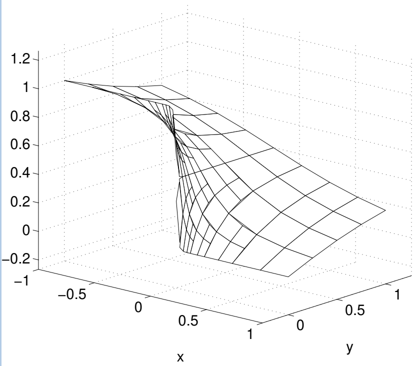
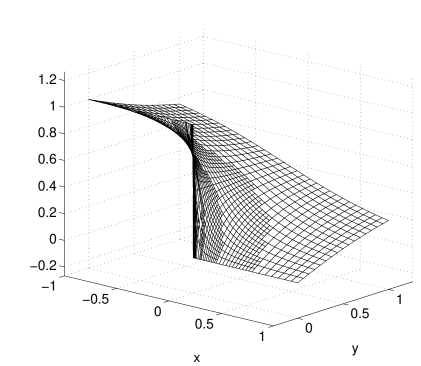
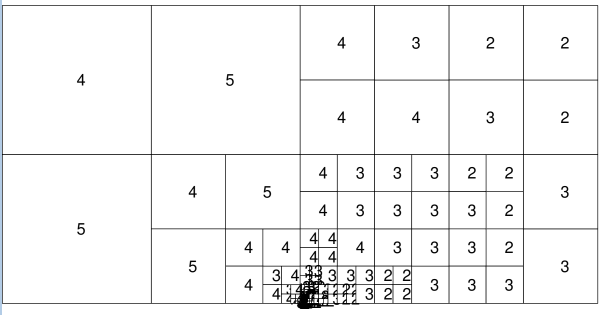




hp-adaptive DG discretisation



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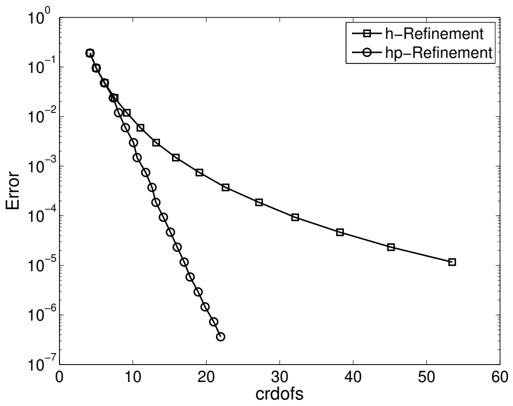

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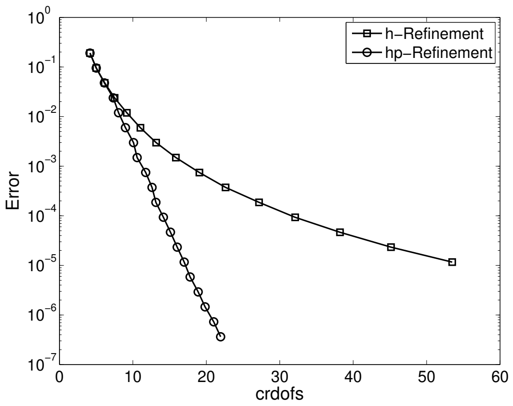

Thank you for listening!