4-lateral matroids induced by 3-configurations

(preprint)

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3-lateral matroids

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We say that $\mathcal{M}_{tri}(\mathcal{C}) = (E, \mathcal{B})$ is a *trilateral matroid*, or *triangular matroid*, or *3-matroid*, which is *induced* by $\mathcal{C}$ if the set of non-trilaterals in $\mathcal{C}$ forms a set of bases $\mathcal{B}$ for a rank-3 matroid.

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This means that $\mathcal{B}$ must satisfy the *basis extension property*: If $X, Y, \in \mathcal{B}$ and $x \in X \setminus Y$, then there exists $y \in Y \setminus X$ such that $X - x \cup y \in \mathcal{B}$.

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We may use either $E = \mathcal{P}$ or $E = \mathcal{L}$ as the ground set of the matroid, since there is a one-to-one correspondence between the set of point triples and the set of line triples. Later, when we enlarge our scope to consider 4-lateral matroids induced by 3-configurations, we will only use $E = \mathcal{L}$.

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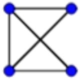

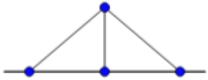

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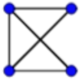

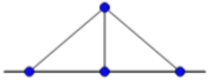

The enumeration of the 3-configurations inducing 3-lateral matroids was conducted by Raney for $7 \leq n \leq 14$, and then extended by Al-Azemi and Raney (21) to $15 \leq n \leq 18$, while also correcting a computational error in the former paper.

Examples of 3-lateral matroids

In the Fano 7_3-configuration every point triple $\{p_1, p_2, p_3\}$ gives a trilateral, so $\mathcal{B} = \emptyset$. So we could deem its 3-lateral matroid to be the uniform matroid $U_{7,2}$. As this is a rank-2 matroid, we say that this is an exceptional case.

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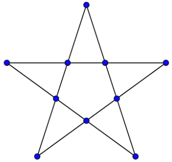

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Another result is that if each point of $\mathcal{C}$ is involved in three triangles, with no pair of points sharing more than one triangle, then $\mathcal{C}$ induces a 3-lateral matroid $\mathcal{M}_{tri}(\mathcal{C})$ which is isomorphic to $\mathcal{C}$.

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For instance, the Coxeter 12_3-configuration satisfies this condition, as does the cyclic configuration $Cyc(n, 4)$ for $n \geq 13$ having Golomb ruler 0 1 4.

Finally, the Cremona-Richmond 15_3-configuration is the smallest 3-configuration which is trilateral-free. So the 3-lateral matroid it induces is the uniform matroid $U_{15,3}$.

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Let $\mathcal{C} = (\mathcal{P}, \mathcal{L})$ be a 3-configuration. We determine the conditions under which $\mathcal{M}_{quad} = (E, \mathcal{B})$ is a rank-4 matroid, where $E = \mathcal{L}$ and $\mathcal{B}$ consists of the line quadruples $\{l_1, l_2, l_3, l_4\} \subseteq E$ which are not 4-laterals in $\mathcal{C}$.

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A fundamental obstruction is that no two 4-laterals may share exactly three lines. Why is this so?

4-lateral matroids

Suppose $\{l_1, l_2, l_3, l_4\}$ and $\{l_1, l_2, l_4, l_5\}$ are 4-laterals which share the lines l_1, l_2, and l_4.

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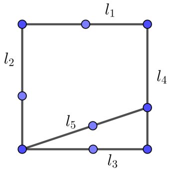

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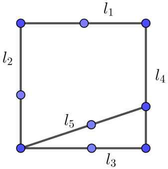

Let l_0 be any other line which does not form a 4-lateral with l_1, l_2 and l_4. Set $X = \{l_0, l_1, l_2, l_4\}$ and $Y = \{l_1, l_2, l_3, l_5\}$ (Assume that Y is not a 4-lateral.)

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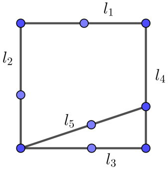

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Then $X \setminus Y = \{l_0, l_4\}$. Take $l_0 \in X \setminus Y$. Then $X - l_0 = \{l_1, l_2, l_4\}$. Note $Y \setminus X = \{l_3, l_5\}$.

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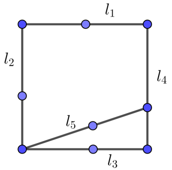

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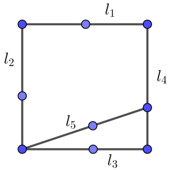

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So in a 3-configuration which induces a 4-lateral matroid, any two distinct 4-laterals may share at most two lines.

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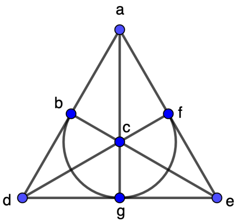

0	1	2	3	4	5	6
a	b	c	d	e	f	g
b	c	d	e	f	g	a
d	e	f	g	a	b	c

When one deletes any point and the three lines incident to it, one obtains a 4-lateral. It follows that the seven 4-laterals in the Fano configuration are of the form $\{k, k+1, k+2, k+4\}$, where $0 \leq k \leq 6$ and all of the numbers are given modulo 7. This 4-lateral shares two lines with the 4-lateral $\{k+1, k+2, k+3, k+5\}$ as well as two lines with the 4-lateral $\{k+3, k+4, k+5, k\}$.

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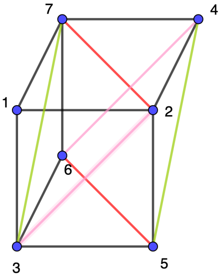

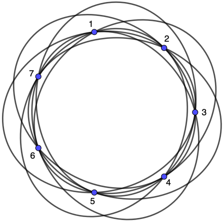

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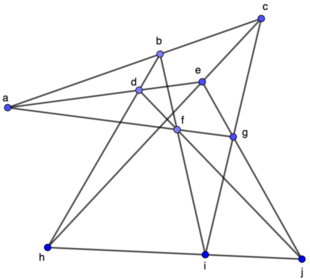

1	2	3	4	5	6	7	8	9	0
a	a	a	b	b	c	c	d	e	h
b	d	f	d	f	e	g	f	g	i
c	e	g	h	i	h	i	j	j	j

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Each point is in perspective to two triangles. For instance, regard point a. It is in perspective to triangles $b\,d\,f$ and $c\,e\,g$. These lead to two 4-laterals $\{1, 3, 4, 8\}$ and $\{1, 3, 6, 9\}$.

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So the associated 4-lateral matroid whose ground set consists of the ten lines of the configuration has $\binom{10}{4} - 20 = 190$ bases.

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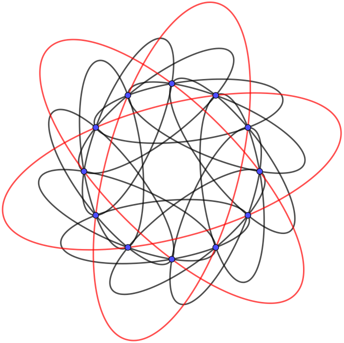
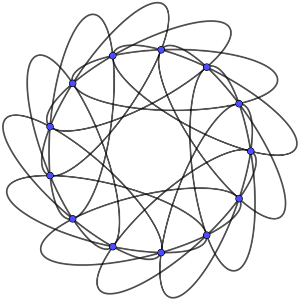
$Cyc(12,3)$ has three additional 4-laterals: $\{1,4,7,10\}$, $\{2,5,8,11\}$, and $\{3,6,9,12\}$.

$Cyc(n, 3)$





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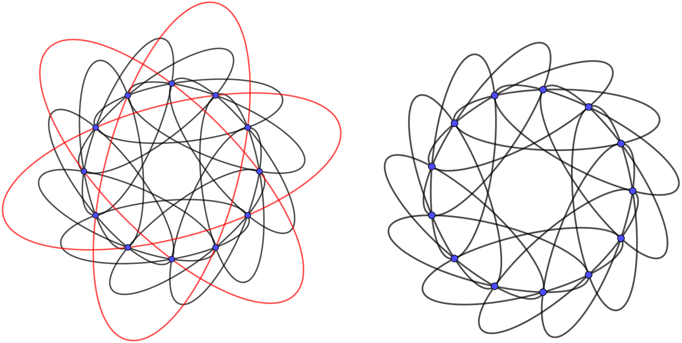
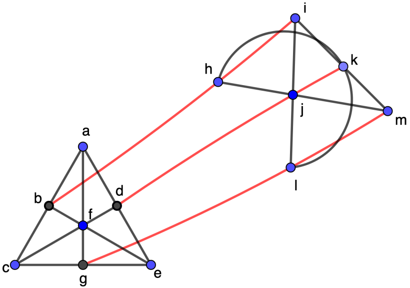


$M_{quad}(Cyc(12, 3))$ and $M_{quad}(Cyc(13, 3))$

Another 13_3-configuration



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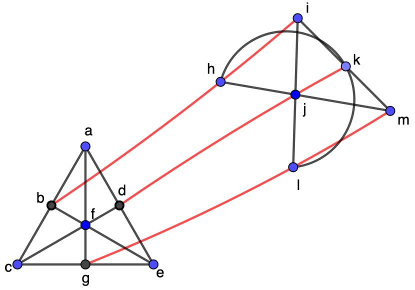

Obtainable from two disjoint copies of Fano. Delete line from the first copy, delete point and lines incident to it from the second copy.

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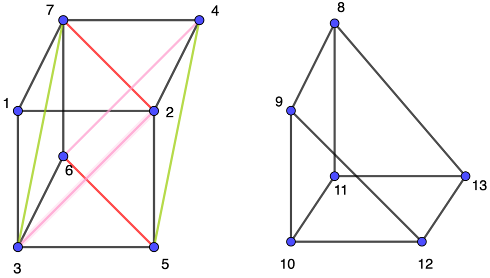

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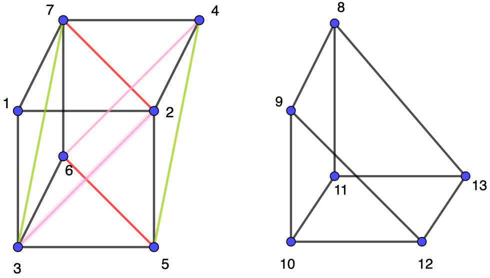

The 4-matroid of this configuration

3-configurations inducing matroids

n	$\mathcal{M}_{tri}$	$\mathcal{M}_{quad}$
7	0	1
8	0	0
9	0	0
10	1	1
11	0	0
12	1	1
13	1	2
14	4	13

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13	1	2
14	4	13

Some of the 4-lateral matroids induced by the 13 14_3-configurations are isomorphic.