Eulerian and Even-Face Ribbon Graph Minors

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Outline

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1. Ribbon graphs
2. Ribbon graph minors
3. Motivation
4. Definitions
5. Characterizing even-face and Eulerian ribbon graphs
6. Characterizing plane Eulerian and plane bipartite ribbon graphs

Ribbon graphs

- Cellularly embedded graphs can be realized as ribbon graphs.

Ribbon graphs

Definition ([1])

A *ribbon graph* $\mathbf{G} = (V(\mathbf{G}), E(\mathbf{G}))$ is a surface with boundary represented as the union of two sets of topological discs: a set $V(\mathbf{G})$ of vertices, and a set $E(\mathbf{G})$ of edges such that

- the vertices and edges intersect in disjoint line segments (we call such line segments as *common-line-segments* since they belong to boundaries of both vertex discs and edge discs);
- each common-line-segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two common-line-segments.

Ribbon graphs

Ribbon graphs have certain advantages over cellularly embedded graphs.

- The deletion of edges or vertices of a ribbon graph results in another ribbon graph, whereas deleting an edge or a vertex of a cellularly embedded graph in a surface Σ may not result in a cellularly embedded graph in the surface Σ.
- Geometric duals have a particularly neat description in the language of ribbon graphs.

- We can regard a ribbon graph $\mathbf{G} = (V(\mathbf{G}), E(\mathbf{G}))$ as a punctured surface. Filling in the punctures by a set of discs denoted $V(\mathbf{G}^*)$, we obtain a surface without boundary. The *geometric dual* $\mathbf{G}^*$ of $\mathbf{G}$ comes out as the ribbon graph $(V(\mathbf{G}^*), E(\mathbf{G}))$ when all the original vertex open discs of $\mathbf{G}$ are removed.

Outline

Ribbon graph minors

- Ribbon graph minors is introduced by Moffatt recently in [2].
- Contracting loops is necessary.

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Motivation

Conjecture (Moffatt JGT 2016)

Every ribbon graph minor-closed family of ribbon graphs can be characterized by a finite set of excluded ribbon graph minors.

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- An analogue of the graph minor theorem (Robertson-Seymour Theorem)

Motivation

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Graphs are well-quasi-ordered under the graph minor relation.

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- A quasi-ordering on a set is a *well-quasi-ordering* if it contains neither an infinite antichain nor an infinite decreasing sequence.

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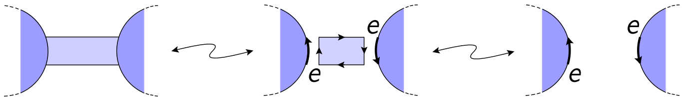
- A quasi-ordering on a set is a *well-quasi-ordering* if it contains neither an infinite antichain nor an infinite decreasing sequence.
- An *antichain* is a subset with the property that any two elements are incomparable.

Arrow-marked ribbon graphs



$\mathbf{G} = (\mathbf{G}\vec{-}\{e\})\vec{+}\{e\}$ $\mathbf{G}\vec{-}\{e\}$

Figure: $\mathbf{G} = (\mathbf{G}\vec{-}\{e\})\vec{+}\{e\}$ and $\mathbf{G}\vec{-}\{e\}$

- Let $D \subset R_2$ be a link diagram. The *ribbon graph of D* (or the *All-A ribbon graph of D*), denoted $\mathbb{A}(D)$ is formed as follows.
  - Assign a unique label to each crossing of D:
  - Take an arrow marked A-smoothing for each crossing of D.

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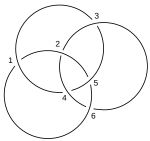
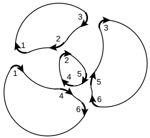
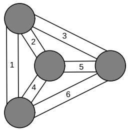
  - Assign a unique label to each crossing of D:
  - Take an arrow marked A-smoothing for each crossing of D.
- We say that a ribbon graph **G** represents a link diagram, or is the ribbon graph of a link diagram, if $\mathbf{G} = \mathbb{A}(D)$ for some link diagram D.

Representing Link Diagrams by Ribbon Graph

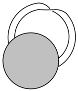




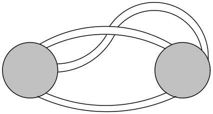


Theorem (Moffatt JGT 2016)

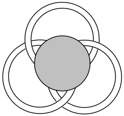
A ribbon graph represents a link diagram if and only if it contains no ribbon graph minor equivalent to $B_{\bar{1}}$, B_3, or θ_t.



$B_{\bar{1}}$



θ_t



B_3

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Theorem (Chudnovsky et al. JCTB 2016)

A bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor.

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- A contraction of a vertex u with a vertex v is called *admissible* if u and v have a common neighbor, and at least one of these common neighbors, say w, is such that the path (u, w, v) is a part of a peripheral cycle.

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- A contraction of a vertex u with a vertex v is called *admissible* if u and v have a common neighbor, and at least one of these common neighbors, say w, is such that the path (u, w, v) is a part of a peripheral cycle.
- A bipartite analog of Wagner's theorem

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- An *even-face graph* is a cellularly embedded graph with no odd degree faces.
- If a bipartite graph cellularly embed into a surface, then it is an even-face graph.
- The set of underlying graphs of all even-face graphs includes all bipartite graphs.

Outline

Partial duals

Let $\mathbf{G}$ be a ribbon graph and $A \subseteq E(\mathbf{G})$. Then the *partial dual* of $\mathbf{G}$ with respect to A, denoted by $\mathbf{G}^A$, is given by

$$\mathbf{G}^A := \left(\mathbf{G} \vec{-} A^c\right)^* \vec{+} A^c.$$

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Proposition (Chmutov, JCTB 2009)

Let $\mathbf{G}$ be a ribbon graph and $A, B \subset E(\mathbf{G})$. Then

- $\mathbf{G}^{\varnothing} = \mathbf{G}$;
- $\mathbf{G}^{E(\mathbf{G})} = \mathbf{G}^*$;
- $\mathbf{G}^{A \cup B} = \left(\mathbf{G}^A\right)^{B \setminus A}$;
- $\mathbf{G}$ *is orientable if only if* $\mathbf{G}^A$ *is orientable;*
- *partial duality acts disjointly on components.*

- If u_1 and u_2 are the (not necessarily distinct) vertices incident to e, then $\mathbf{G}/e$ denotes the ribbon graph obtained as follows: consider the boundary component(s) of $e \cup u_1 \cup u_2$ as curves on $\mathbf{G}$. For each resulting curve, attach a disc (which will form a vertex of $\mathbf{G}/e$) by identifying its boundary component with the curve. Delete e, u_1 and u_2 from the resulting complex, to get the ribbon graph $\mathbf{G}/e$. We say $\mathbf{G}/e$ is obtained from $\mathbf{G}$ by *contracting* e.

- Notice that each edge in a ribbon graph contains exactly two line segments not lying on the boundary of its end vertices (vertex), we call such line segments *edge-line-segments* of the edge.

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- A ribbon graph **G** is an *even-face ribbon graph* if the degree of its every boundary component is even.
- Any cellularly embedded bipartite graph is equivalent to an even-face ribbon graph, and if an even-face ribbon graph is plane then it is bipartite.

- The *distance* of two vertex-line-segments (resp. two edge-line-segments) lying on a same boundary component is the minimum number of edge-line-segments lying between the two vertex-line-segments (resp. two edge-line-segments).

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- The *dual-distance* of two vertex-line-segments (resp. two line-segments) lying on the boundary of a vertex disc is the minimum number of line-segments lying between the two vertex-line-segments (resp. two line-segments).

Definitions

evenly splitting a face:

1. choose two vertex-line-segments with even distance;
2. separately place an e colored arrow on the two vertex-line-segments such that the directions of the two e colored arrows are coherent with a direction of traveling around the boundary component including the two vertex-line-segments;
3. $\mathbf{G}\vec{+}\{e\}$;
4. $\left(\mathbf{G}\vec{+}\{e\}\right)/e$.

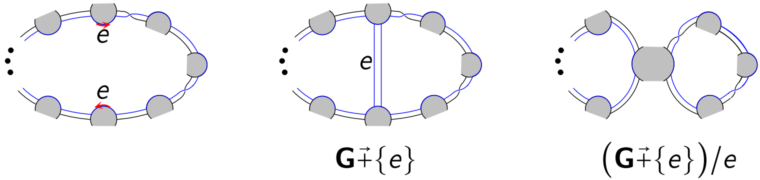

Figure: Even splitting face.

Definitions

evenly splitting a vertex:

1. choose two vertex-line-segments with even dual-distance;
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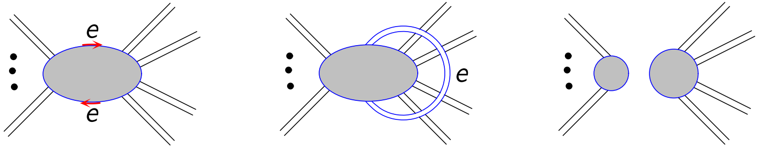

$\mathbf{G}\vec{+}\{e\}$ $\qquad$ $\big(\mathbf{G}\vec{+}\{e\}\big)/e$

Figure: Even splitting vertex.

- We say that $\mathbf{G}/e$ is *proper* if and only if e is not an orientable loop with odd dual distance common-line-segments, and $\mathbf{G} - e$ is *proper* if and only if $\mathbf{G}^*/e^*$ is proper.

Definition

A ribbon graph $\mathbf{H}$ is an *even-face minor* of a ribbon graph $\mathbf{G}$ if there is a sequence of ribbon graphs $\mathbf{G} = \mathbf{G}_0, \mathbf{G}_1, \cdots, \mathbf{G}_t = \mathbf{H}$ such that for each i, $\mathbf{G}_{i+1}$ is obtained from $\mathbf{G}_i$ by a proper edge deletion, component deletion, or evenly splitting a face.

Definitions

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A ribbon graph $\mathbf{H}$ is an *Eulerian minor* of a ribbon graph $\mathbf{G}$ if there is a sequence of ribbon graphs $\mathbf{G} = \mathbf{G}_0, \mathbf{G}_1, \cdots, \mathbf{G}_s = \mathbf{H}$ such that for each i, $\mathbf{G}_{i+1}$ is obtained from $\mathbf{G}_i$ by a proper edge contraction, component deletion, or evenly splitting a vertex.

Lemma

A ribbon graph $\mathbf{H}$ *is an Eulerian minor of a ribbon graph* $\mathbf{G}$ *if and only if* $\mathbf{H}^*$ *is an even-face minor of* $\mathbf{G}^*$*.*

Outline

Characterizing even-face and Eulerian ribbon graphs

Lemma

Let G be a cellularly embedded graph. Then G is Eulerian if and only if its geometric dual G^* is an even-face graph.

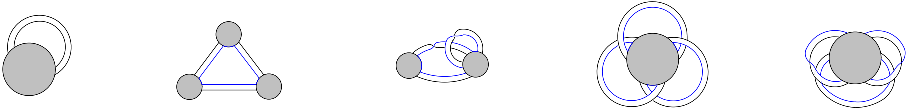
Lemma

The set of even-face ribbon graphs is even-face minor closed.

Characterizing even-face ribbon graphs

Theorem (Metsidik and Jin DM 2020)

A ribbon graph is an even-face ribbon graph if and only if it contains no even-face minor equivalent to $\mathbf{B}_1, \mathbf{C}_3, \mathbf{Q}, \mathbf{B}_3$ *or* $\mathbf{R}$.



$\mathbf{B}_1$ $\quad$ $\mathbf{C}_3$ $\quad$ $\mathbf{Q}$ $\quad$ $\mathbf{B}_3$ $\quad$ $\mathbf{R}$

Characterizing Eulerian ribbon graphs

Lemma

The set of Eulerian ribbon graphs is Eulerian-minor closed.

Theorem

A ribbon graph is Eulerian if and only if it contains no Eulerian-minor equivalent to $\mathbf{B}_1^*, \mathbf{C}_3^*, \mathbf{Q}^*, \mathbf{B}_3^*$ *or* $\mathbf{R}^*$.

Outline

Genus of Ribbon graphs

The *genus* of a ribbon graph $\mathbf{G}$, denoted by $g(\mathbf{G})$, is calculated by Euler's formula as in the following.

$$g(\mathbf{G}) = \begin{cases} k(\mathbf{G}) - \frac{1}{2}\big(v(\mathbf{G}) - e(\mathbf{G}) + p(\mathbf{G})\big), & \text{if } \mathbf{G} \text{ is orientable;} \\ 2k(\mathbf{G}) - \big(v(\mathbf{G}) - e(\mathbf{G}) + p(\mathbf{G})\big), & \text{if } \mathbf{G} \text{ is non-orientable,} \end{cases}$$

where $v(\mathbf{G}) = |V(\mathbf{G})|, e(\mathbf{G}) = |E(\mathbf{G})|$, $k(\mathbf{G})$ and $p(\mathbf{G})$ are the numbers of the connected components and boundary components of $\mathbf{G}$, respectively. In particular, $\mathbf{G}$ is *plane* if $g(\mathbf{G}) = 0$.

Characterizing plane Eulerian and plane bipartite ribbon graphs

Lemma

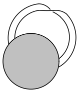
Let $\mathbf{H}$ *be an even-face minor of a ribbon graph* $\mathbf{G}$*. Then* $g(\mathbf{H}) \leq g(\mathbf{G})$*.*

Lemma

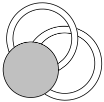
Let $\mathbf{H}$ *be an Eulerian-minor of a ribbon graph* $\mathbf{G}$*. Then* $g(\mathbf{H}) \leq g(\mathbf{G})$*.*

Theorem (Metsidik and Jin DM 2020)

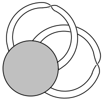
An Eulerian ribbon graph is plane if and only if it contains no Eulerian minor equivalent to $\mathbf{B}_{\bar{1}}, \mathbf{B}_3 - e, \mathbf{B}_{\bar{3}} - e$ *or* $\mathbf{B}_3$.



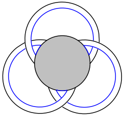
$\mathbf{B}_{\bar{1}}$



$\mathbf{B}_3 - e$



$\mathbf{B}_{\bar{3}} - e$



$\mathbf{B}_3$

Characterizing plane Eulerian and plane bipartite ribbon graphs

Theorem

An even-face ribbon graph is plane if and only if it contains no even-face minor equivalent to $\mathbf{B}_{\bar{1}}, \mathbf{B}_3 - e, (\mathbf{B}_3 - e)^*$ *or* $\mathbf{B}_3^*$.

References I

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Thank you for listening!