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Materials with memory: an overview on admissible kernels in the integro-differential model equations

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- **The Viscoelastic Material Model**

regular, singular . . . weaker formulations . . . aging effects

- **Existence & Uniqueness of Solutions**
- **. . . . . . also in Magneto-Viscoelastic Materials**

bibliography :

– **G.Amendola, M.Fabrizio, J.M. Golden**, (book) Springer, 2012;
– **S.C. V.Valente, G.Vergara Caffarelli**, 2011, 2012, 2013, 2014;
– **S.C., M. Chipot, V.Valente, G.Vergara Caffarelli**, 2017, 2019, preprint 2021
– **S.C., C. Giorgi**, Intech 2016, preprint 2021
– **G.Amendola, S.C., A.Manes**, 2010
– **G.Amendola, S.C., J.M. Golden, A.Manes**, DCDS-B 2014
– **G.Amendola, S.C.**, 2004
– **S.C.,** 2005, 2010, 2011a, 2011b, 2015, 2017, 2019.

General Framework of the Problem

the viscoelastic material model

- $\Omega \subset \mathbb{R}^3$ connected set: the body configuration;
- viscoelastic body $\Longrightarrow$ shape change of Ω according to linear viscoelasticity;
- memory effects $\Longrightarrow$ the deformation depends on time via present and past times;
- passive environment $\Longrightarrow$ environment's status is not affected by the body's one.

... in addition ...

- no space dependence i.e. $\mathbf{x}$-independence

key references

M. Fabrizio, A. Morro, 1992; G. Gentili, 2001

Quantities of Interest

$\mathbf{E} = \mathbf{E}(t)$	*strain tensor*
$\mathbf{T} = \mathbf{T}(t)$	*stress tensor*
$\mathbf{G} = \mathbf{G}(t)$	*relaxation modulus*
$\mathbf{G}_0 = \mathbf{G}(0)$	*initial relaxation modulus*

furthermore ...... constitutive assumptions

$$\mathbf{T}(t) = \int_0^\infty \mathbf{G}(\tau)\ \dot{\mathbf{E}}(t-\tau)\ d\tau \quad , \quad \mathbf{G}(t) = \mathbf{G}_0 + \int_0^t \dot{\mathbf{G}}(s)\ ds$$

or equivalently, when $\mathbf{E}^t(\tau)$ denotes the **strain past history**

$$\mathbf{T}(t) = \mathbf{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbf{G}}(\tau)\ \mathbf{E}^t(\tau)\ d\tau \quad , \quad \mathbf{E}^t(\tau) := \mathbf{E}(t-\tau)$$

Regular Memory Kernel

1-dimensional Viscoelastic Problem

Classical Problem **C.Dafermos 1970**

$$u_{tt} = G(0)u_{xx} + \int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau + f$$

$$u(\cdot,0) = u_0, \;\; u_t(\cdot,0) = u_1 \text{ in } \Omega\,; \;\; u = 0 \text{ on } \Sigma = \partial\Omega \times (0,T)$$

u displacement and f external force (history of the material included)

$$\dot{G} \in L^1(I\!R^+) \;\;,\;\; G(t) = G_0 + \int_0^t \dot{G}(s)\; ds\;,\; G(\infty) = \lim_{t\to\infty} G(t)$$

Hence, G enjoys the fading memory property

$$\forall \epsilon > 0\, \exists\, \tilde{a} = a\,(\epsilon, E^t) \in I\!R^+ s.t.\; \forall a > \tilde{a}, \left| \int_0^\infty \dot{G}(s+a)E^t(s)\; ds \right| < \epsilon$$

Generalisations

weaker assumptions on the relaxation modulus G

- **Singular Memory Kernel i.e. unbounded relaxation modulus G**

  S.C, V.Valente, G.Vergara Caffarelli 2013

  require $\boxed{G \in L^1(0,T), \forall T \in \mathbb{R}}$

  ... hence it follows $\lim_{t \to 0^+} G(t) = +\infty$

- **Weak regular relaxation modulus G**

  S.C, M. Chipot, V.Valente, G.Vergara Caffarelli 2019 Assume

$$\boxed{\forall\ t \in (0,\infty), \quad G(t) > 0, \quad G \ \text{ is non-increasing and convex.}}$$

removing the classical assumptions, $\dot{G} \notin L^1(\boldsymbol{R}^+)$ Hence,

$$\boxed{\int_0^t \dot{G}(s)\, ds \ \text{ not defined } \ , \ G \in C^0[0,T] \ \ \forall T \in \mathbb{R}}$$

Further Generalisations

... and weaker assumptions

- **relaxation modulus G admitting a jump discontinuity**

  S.C, M. Chipot, V.Valente, G.Vergara Caffarelli 2021 in progress

  - $G(s) > 0 \quad , \quad 0 < s < \infty$
  - G **monotonic not increasing function;**
  - $G_\infty := \lim_{s\to\infty} G(s) \geq 0$**;**
  - $\displaystyle\int_0^\infty (G(s) - G_\infty)\, ds < \infty$**, that is** $G(s) - G_\infty \in L^1(0, \infty)$**.**

... and "aging" effects

- **Time dependent relaxation modulus G**

  [M.Conti, V.Danese, C.Giorgi, V.Pata, 2018, S.C., C.Giorgi 2016]

  The relaxation modulus G depends on the time variables t, τ **not only** trough the difference $t - \tau$ but

  $\boxed{G := G(t, \tau)}$ ... **combine it with magnetisation**

Obtained Results

- solution's existence relies on Free Energy;
- singular viscoelasticity 1D **S.C, V.Valente, G.Vergara Caffarelli 2013**
- singular rigid heat conduction with memory 1D & 3D
  S.C, V.Valente, G.Vergara Caffarelli 2014, S.C 2015
- 1 D singular magneto-viscoelasticity problem;
  S. C., M.Chipot, V. Valente, G. Vergara Caffarelli, NONRWA 2017
- 3 D singular viscoelasticity problem;
  S. C., AAPP 2019
- a weakly regular relaxation modulus,
  S. C., M.Chipot, V. Valente, G. Vergara Caffarelli, CAIM 2019

... in progress ...

- 1 D magneto-viscoelasticity problem with *aging*
  S.C., C.Giorgi, preprint 2021
- 1 D viscoelasticity problem with *weaker possible* regularity on G
  S.C., M.Chipot, V. Valente, G. Vergara Caffarelli, in progress 2021

Magneto-viscoelasticity problem

combine *non classical* **viscoelasticity**

- **Singular viscoelasticity** i.e. unbounded relaxation modulus G existence & uniqueness (1-dimensional problem)

  S.C, V.Valente, G.Vergara Caffarelli 2013
- **Weakly regular relaxation modulus G**

  S.C, M. Chipot, V.Valente, G.Vergara Caffarelli 2019
- **Relaxation modulus $G = G(t, \tau)$ "aging" effects**

  S.C, C. Giorgi, 2016, M.Conti, V.Danese, C.Giorgi, V.Pata, 2018

with magnetisation

Singular Memory Kernel

1-dimensional Viscoelastic Problem

Idea S.C, V.Valente, G.Vergara Caffarelli 2013

- relax the assumptions on the relaxation modulus G:
  require $\boxed{G \in L^1(0,T), \forall T \in \mathbb{R}}$
  $\dots$ hence it follows $\lim_{t \to 0^+} G(t) = +\infty$
- construct regular approximated problems:
- find approximated solutions u^ε;
- show $\exists\ u := \lim_{\varepsilon \to 0} u^\varepsilon$
- recognize uniqueness of u solution to the singular problem.

Approximation Strategy: Idea

Approximated Problems

- *approximated* problems $0 < \varepsilon < 1, G^\varepsilon(\cdot) = G(\varepsilon + \cdot)$

  translated relaxation modulus

Remark: ∃! according to Dafermos [68, 70]

$$u^\varepsilon_{tt} = G^\varepsilon(0)\, u^\varepsilon_{xx} + \int_0^t \dot{G}^\varepsilon(t-\tau)\, u^\varepsilon_{xx}(\tau)\, d\tau + f \ \ , \text{ i.c. } \ \& \ \text{ b.c.}$$

- *integral* formulation of the problems

$$u^\varepsilon(t) = \int_0^t K^\varepsilon(t-\tau) u^\varepsilon_{xx}(\tau) d\tau + u_1 t + u_0 + \int_0^t d\tau \int_0^\tau f(\xi) d\xi \,,$$

$K(\xi) := \int_0^\xi G(\tau) d\tau$ well defined since $G \in L^1(0,T), \forall T \in \mathbb{R}$

K **integrated relaxation modulus**

Existence of the limit Solution

Theorem Given u^ε solution to the integral problem P^ε

$$P^\varepsilon : u^\varepsilon(t) = \int_0^t K^\varepsilon(t-\tau) u^\varepsilon_{xx}(\tau) d\tau + u_1 t + u_0 + \int_0^t d\tau \int_0^\tau f(\xi) d\xi \, ,$$

then

$$\exists \; u(t) = \lim_{\varepsilon \longrightarrow 0} u^\varepsilon(t) \quad \text{in} \quad L^2(\mathcal{Q}) \; , \; \mathcal{Q} = \Omega \times (0,T).$$

Proof (Idea):

- weak formulation, on introduction of test functions $\varphi \in H^1(\Omega \times (0,T))$ s.t. $\varphi = 0$ on $\partial\Omega$;
- consider separately the terms without ε;
- the terms with u^ε & K^ε
- prove convergence via Lebesque's Theorem.

Solution uniqueness

Theorem The integral problem

$$u(t) = \int_0^t K(t-\tau) u_{xx}(\tau) d\tau + u_1 t + u_0 + \int_0^t d\tau \int_0^\tau f(\xi) d\xi \,,$$

admits a unique weak solution.

Proof (by contradiction)

- Let $w := v - \tilde{v}, v \neq \tilde{v}$, where $v, \tilde{v}$ are weak solutions $\Longrightarrow w$ satisfies
$$\iint_R w(t)\psi(x,t) dx dt = \iint_R \psi_{xx}(x,t) \int_0^t K(t-\tau) w(\tau) d\tau dx dt$$
- recall the b.c.s $w(0,t) = w(\pi, t) = 0 \; \forall t \in (0, T)$, choose the test function $\psi(x,t)$ and $w_m(x,t)$ via
$$\psi(x,t) = \varphi(t) \sin(mx) \;, \quad w_m(x,t) = \sum_{m=0}^{\infty} \alpha_m(t) \sin(mx);$$
- orthogonality combined with Gronwall's Lemma implies $\forall m \in \mathbb{N}$,
$$\int_0^T \varphi(t) \left[\alpha_m(t) - \int_0^t K(t-\tau) m^2 \alpha_m(t) d\tau \right] dt = 0 \implies \boxed{\alpha_m(t) = 0}.$$

□

Weak regular relaxation modulus

S. C., M.Chipot, V. Valente, G. Vergara Caffarelli, CAIM 2019

Assume

$\forall\ t \in (0,\infty), \quad G(t) > 0, \quad G$ **is non-increasing and convex.**

removing the classical assumptions, $\dot{G} \notin L^1(\mathbb{R}^+)$ Hence,

$$\int_0^t \dot{G}(s)\, ds \ \textbf{not defined}\ , \ G \in C^0[0,T] \ \forall T \in \mathbb{R}$$

$G \in L^1(0,T), \forall T \in \mathbb{R} \Longrightarrow K(\xi) := \int_0^\xi G(\tau) d\tau$ **well defined**

where K **integrated relaxation modulus**. Let

$$G^\varepsilon(t) := \int_{t-\varepsilon}^{t+\varepsilon} \rho\left(\frac{t-\tau}{\varepsilon}\right) \frac{1}{\varepsilon} G(\varepsilon+\tau) d\tau\ ,\ 0 < \varepsilon < 1 \qquad (1)$$

Existence of weak solutions

$$P^{\varepsilon_h}: \; u^{\varepsilon_h}(t) = \int_0^t K^{\varepsilon_h}(t-\tau) u_{xx}^{\varepsilon_h}(\tau) d\tau + u_1 t + u_0 + \int_0^t d\tau \int_0^\tau f(\xi) d\xi \,,$$

where $\{\varepsilon_h\}, h \in \mathbb{N}, \lim_{h\to\infty} \varepsilon_h = 0$

Theorem

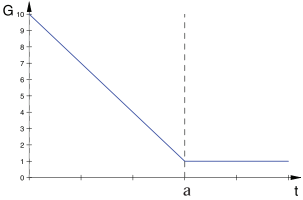
$$\boxed{\begin{array}{l} u^{\varepsilon_h} \longrightarrow \; u \;\text{ weakly }\; \text{in } H^1(D) \;\text{ as } \varepsilon_h \to 0 \\ u^{\varepsilon_h} \longrightarrow \; u \;\text{ strongly }\; \text{in } L^2(D) \;\text{ as } \varepsilon_h \to 0 \end{array}}$$

The proof is in

S. C., M.Chipot, V. Valente, G. Vergara Caffarelli, *On weak regularity requirements of the relaxation modulus in viscoelasticity*, Comm.s in Applied and Industrial Mathematics (CAIM), **10** (1), (2019), 78-87, 2019

A notable example

$$\text{G(s)}=\begin{cases} \dfrac{G_\infty - G_0}{a} s + G_0 & 0 \le s \le a \\ G_\infty & s \ge a \end{cases}$$



Relaxation function

The evolution equation

$$u_{tt} = G(0)u_{xx} + \int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau + f,$$

now, on substitution of the given relaxation function G,

$$u_{tt} = G(0)u_{xx} + [G_\infty - G_0]\int_{\tilde{t}}^t \frac{1}{a}u_{xx}(s)ds + f \ , \ \tilde{t} := \max\{0, t-a\} \ .$$

The limit $a \to 0$, on application of the mean value theorem, gives

the wave equation: $\boxed{u_{tt} = G_\infty u_{xx} + f}$.

General Framework of the Problem

The magnetic model

- $\Omega \subset \mathbb{R}^3$ connected set: the body configuration;
- magnetic body $\Longrightarrow$ magnetization of Ω changes according to the Landau Lifshitz equation:

in Gilbert form, where $\mathbf{m}$ represents the magnetization vector

$$\gamma^{-1}\mathbf{m}_t - \mathbf{m} \times (a\,\Delta\mathbf{m} - \mathbf{m}_t) = 0 \quad , |\mathbf{m}| = 1, \; \gamma, a \in \mathbb{R}^+$$

references

- **M. Bertsch, P. Podio Guidugli and V. Valente, 2001**
- **T. L. Gilbert, 1955**

Magneto-Viscoelasticity Model

- interaction between the two different effects;
- model equations $\rho, \gamma, a \in I\!R^+$ given — 3 dimensional

$$\begin{cases} \gamma^{-1}\dot{\mathbf{m}} - \mathbf{m} \times (a\mathbf{\Delta m} - \dot{\mathbf{m}} - \mathbb{L}\mathbf{m} \otimes \nabla\mathbf{u}) = 0 \\ \rho\ddot{\mathbf{u}} - \operatorname{div}\left(\mathbb{G}(0)\nabla\mathbf{u} + \int_0^t \dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}(\tau)d\tau + \frac{1}{2}\mathbb{L}\mathbf{m} \otimes \mathbf{m}\right) = \mathbf{f} \end{cases}$$

- model equations $0 < \delta \ll 1$, — 1 dimensional

$$\begin{cases} u_{tt} - G(0)u_{xx} - \int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau - \frac{\lambda}{2}\Big(\Lambda(\mathbf{m}) \cdot \mathbf{m}\Big)_x = f\,, \\ \mathbf{m}_t + \mathbf{m}\frac{|\mathbf{m}|^2 - 1}{\delta} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0\,, \end{cases}$$

references

- **S. C., V. Valente, G. Vergara Caffarelli, D.C.D.S. – S 2012**
- **S. C., V. Valente, G. Vergara Caffarelli, Appl. Anal. 2011**

The evolution problem 3-dim

$$\begin{cases} \gamma^{-1}\dot{\mathbf{m}} - \mathbf{m} \times (a\Delta\mathbf{m} - \dot{\mathbf{m}} - \mathbb{L}\mathbf{m} \otimes \nabla\mathbf{u}) = 0 \\ \rho\ddot{\mathbf{u}} - \operatorname{div}\left(\mathbb{G}(0)\nabla\mathbf{u} + \int_0^t \dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}(\tau)d\tau + \frac{1}{2}\mathbb{L}\mathbf{m} \otimes \mathbf{m}\right) = \mathbf{f} \end{cases} \tag{1}$$

initial

$$\mathbf{m}(0) = \mathbf{m}^0, \ |\mathbf{m}^0| = 1, \quad \mathbf{u}(0) = \mathbf{u}^0, \quad \dot{\mathbf{u}}(0) = \mathbf{u}^1 \tag{2}$$

and boundary conditions

$$\mathbf{m}_\nu|_{\partial\Omega} = 0, \quad \mathbf{u}|_{\partial\Omega} = 0. \tag{3}$$

The evolution problem 1-dim

$$\begin{cases} u_{tt} - G(0)u_{xx} - \displaystyle\int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau - \frac{\lambda}{2}\left(\Lambda(\mathbf{m})\cdot\mathbf{m}\right)_x = f\,, \\ \mathbf{m}_t + \mathbf{m}\dfrac{|\mathbf{m}|^2-1}{\delta} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0\,, \end{cases}$$

initial

$$u(\cdot,0) = u_0, \;\; u_t(\cdot,0) = u_1, \quad \mathbf{m}(\cdot,0) = \mathbf{m}_0, \quad |\mathbf{m}_0| = 1 \quad \text{in } \Omega$$

and boundary conditions

$$u = 0, \qquad \frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on} \quad \Sigma = \partial\Omega \times (0,T)$$

Quantities of Interest

$\mathbf{u} := \mathbf{u}(\mathbf{x}, t)$ *displacement vector*

$\mathbf{m} := \mathbf{m}(\mathbf{x}, t)$ *magnetization vector*

$\mathbb{G}(s) = \{G_{klmn}(s)\}, \ s \in [0, T]$ *visco-elasticity tensor*

$\mathbb{L} = \{\lambda_{klmn}\}$ *magneto-elasticity tensor*

$\mathbb{E} = \{\epsilon_{lm}\}$ *strain tensor*

$G_{klmn}\epsilon_{kl}(\mathbf{u})\epsilon_{mn}(\mathbf{v}) = \mathbb{G}\,\nabla\mathbf{u} \cdot \nabla\mathbf{v},$

$\{\lambda_{klmn} m_k m_l\} = \mathbb{L}\mathbf{m} \otimes \mathbf{m}$

$\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$\{\lambda_{klmn} m_k \epsilon_{lm}(\mathbf{u})\} = \mathbb{L}\mathbf{m} \otimes \nabla\mathbf{u}$

$\lambda_{klmn} m_k m_l \epsilon_{mn}(\mathbf{u}) = \mathbb{L}\mathbf{m} \otimes \mathbf{m} \cdot \nabla\mathbf{u}$

$\epsilon_{lm}(\mathbf{u}) = \frac{1}{2}(\mathbf{u}_{l,m} + \mathbf{u}_{m,l})$ $\epsilon(\mathbf{u})$ *deformation tensor*

furthemore . . . . . . constitutive assumptions

Constitutive Assumptions

$$E_{\mathrm{ex}}(\mathbf{m}) = \frac{1}{2}\int_{\Omega} a_{ij} m_{k,i} m_{k,j} d\Omega$$ exchange magnetization energy

- $a_{ij} = a_{ji}$ symmetric positive definite matrix
- $a_{ij} = a\,\delta_{ji},\ a \in \mathbb{R}^+$ diagonal matrix (most materials)

$$E_{\mathrm{em}}(\mathbf{m},\mathbf{u}) = \frac{1}{2}\int_{\Omega} \lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{u}) d\Omega$$ magneto-elastic energy

- $\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$;
- $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise;
- $\lambda_1, \lambda_2,\ \lambda_3 \in \mathbb{R}$ constants.

Constitutive Assumptions

$$E_{\mathrm{ve}}(\mathbf{u}) = \frac{1}{2}\int_\Omega G_{klmn}(0)\epsilon_{kl}\epsilon_{mn} d\Omega +$$

$$\frac{1}{2}\int_0^t d\tau \left(\int_\Omega \dot{G}_{klmn}(t-\tau)\epsilon_{kl}(\tau)\epsilon_{mn}(\tau) d\Omega\right)$$ viscoelastic energy

- $G_{klmn} = G_{mnkl} = G_{lkmn}$
- $G_{klmn}\, e_{kl}e_{mn} \geq \beta\, e_{kl}e_{kl}, \qquad \beta > 0,\ e_{kl} = e_{lk}$
- $\dot{G}_{klmn}e_{kl}e_{mn} \leq 0$
- $\ddot{G}_{klmn}e_{kl}e_{mn} \geq 0$

Magneto-Viscoelasticity Model

Results: Regular Case

- 1-dimensional problem[1]:

$$\begin{cases} u_{tt} - G(0)u_{xx} - \int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau - \frac{\lambda}{2}\Big(\Lambda(\mathbf{m})\cdot\mathbf{m}\Big)_x = f\,, \\ \mathbf{m}_t + \mathbf{m}\frac{|\mathbf{m}|^2-1}{\delta} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0 \quad + \text{ i.b.c} \end{cases}$$

existence & uniqueness of the weak solution[1]

- 3-dimensional problem[2]:

$$\begin{cases} \gamma^{-1}\dot{\mathbf{m}} - \mathbf{m}\times(a\Delta\mathbf{m} - \dot{\mathbf{m}} - \mathbb{L}\mathbf{m}\otimes\nabla\mathbf{u}) = 0 \\ \rho\ddot{\mathbf{u}} - \operatorname{div}\left(\mathbb{G}(0)\nabla\mathbf{u} + \int_0^t \dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}(\tau)d\tau + \frac{1}{2}\mathbb{L}\mathbf{m}\otimes\mathbf{m}\right) = \mathbf{f} \quad + \text{ i.b.c} \end{cases}$$

existence of weak solution[2]

(1) **S. C., V. Valente, G. Vergara Caffarelli, Appl. Anal. 2011**

(2) **S. C., V. Valente, G. Vergara Caffarelli, , D.C.D.S. Series S 2012**

Theorem (1-D. pb.)

Assumptions:

- $T > 0$
- small enough (i.e. $\delta < \lambda^{-2} G(T)$)

$\Longrightarrow$ $\exists\,!$ weak solution to the problem (1), (2), (3) such that:

- $u \in C^0([0,T];\, H^1_0(\Omega)) \cap C^1([0,T];\, L^2(\Omega))$
- $\mathbf{m} \in \mathbf{C}^0([0,T]; H^1(\Omega)) \cap \mathbf{L}^2(0,T; H^2(\Omega))$
- $\mathbf{m}_t \in \mathbf{L}^2(0,T; L^2(\Omega))$,

key tools viscoelastic term

- viscoelastic free energy;
- Gronwall's lemma;

provide an *a priori estimate* of the viscoelastic term

$$\boxed{\frac{1}{2}\int_\Omega |\varphi_x|^2 dx + \frac{1}{2}\int_\Omega |\varphi_t|^2 dx \le \alpha e^T C(f, \varphi_0, \varphi_1),\ \alpha, C \in \mathbb{R}^+}$$

Theorem (1-D. pb.)

Proof's Ingredients

- a priori estimates (coupled system)
- via two Lemmas
- (the 2nd provides a uniform a priori estimate)
- fixed point theorem.

Remark: crucial is the **free energy functional**

$$\psi = \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{G}(|\tau - s|)\dot{\mathbf{E}}(s)\cdot\dot{\mathbf{E}}(\tau)ds d\tau$$

Theorem (3-D. pb.)

Assumptions:

- $\mathbf{u}^0 \in H^1_0(\Omega;\mathbb{R}^3)$
- $\mathbf{u}^1 \in L^2(\Omega;\mathbb{R}^3)$
- $\mathbf{m}^0 \in H^1(\Omega;\mathbb{R}^3)$, $|\mathbf{m}^0| = 1$ a.e. in Ω
- $\mathbb{G}(s) \in C^2[0,T]$
- $\mathbf{f} \in L^2(\mathcal{Q};\mathbb{R}^3)$, $\mathcal{Q} = \Omega \times [0,T]$

$\Longrightarrow$ $\exists\ (\mathbf{m},\mathbf{u})$ weak solution to the problem (1), (2), (3) such that:

- $\mathbf{m} \in H^1(\mathcal{Q};\mathbb{R}^3)$ with $|\mathbf{m}| = 1$ a.e. in $\mathcal{Q}$
- $\mathbf{u} \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^3))$ and $\dot{\mathbf{u}} \in L^2(\mathcal{Q};\mathbb{R}^3)$
- $\forall(\mathbf{p},\mathbf{g})$ s.t. $\mathbf{g} \in H^1_0(\mathcal{Q};\mathbb{R}^3)$, $\mathbf{p} \in H^{1,\infty}(\mathcal{Q};\mathbb{R}^3)$, $\mathbf{p}(0) = \mathbf{p}(T) = 0$,

$$\int_{\mathcal{Q}} \left[\gamma^{-1}\dot{\mathbf{m}} \cdot \mathbf{p} + a(\mathbf{m} \times \nabla\mathbf{m}) \cdot \nabla\mathbf{p} + \mathbf{m} \times (\dot{\mathbf{m}} + \mathbb{L}\mathbf{m} \otimes \mathbf{p} \cdot \nabla\mathbf{u})\right] d\Omega dt = 0$$

$$\int_{\mathcal{Q}} [-\rho\,\dot{\mathbf{u}} \cdot \dot{\mathbf{g}} + (\mathbb{G}(0)\nabla\mathbf{u} + \frac{1}{2}\,\mathbb{L}\mathbf{m} \otimes \mathbf{m}) \cdot \nabla\mathbf{g}\,]\, d\Omega dt +$$
$$+ \int_{\mathcal{Q}} \left(\int_0^t \dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}(\tau) \cdot \nabla\mathbf{g}(\tau) d\tau\right) d\Omega dt - \int_{\mathcal{Q}} \mathbf{f} \cdot \mathbf{g}\, d\Omega dt = 0$$

Theorem's Proof

Outline:

- Approximated penalty problem:
  - ▷ introduce a small positive parameter δ and consider in $\mathcal{Q}$ an approximated i.b.v. problem;
  - ▷ consider the related Faedo-Galerkin approximation
- Convergence of the approximate solutions;

Approximated penalty problem

$$\begin{cases} \gamma^{-1}\dot{\mathbf{m}}^\delta \times \mathbf{m}^\delta + \dot{\mathbf{m}}^\delta - a\Delta\mathbf{m}^\delta + \mathbb{L}\mathbf{m}^\delta \otimes \nabla\mathbf{u}^\delta + \delta^{-1}(|\mathbf{m}^\delta|^2 - 1)\mathbf{m}^\delta = 0 \\ \rho\ddot{\mathbf{u}}^\delta - \operatorname{div}\left(\mathbb{G}(0)\nabla\mathbf{u}^\delta + \int_0^t \dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}^\delta(\tau)d\tau + \frac{1}{2}\mathbb{L}\mathbf{m}^\delta \otimes \mathbf{m}^\delta\right) = \mathbf{f} \end{cases}$$

i.b.c.

$$\mathbf{u}^\delta(\cdot,0) = \mathbf{u}^0, \quad \dot{\mathbf{u}}^\delta(\cdot,0) = \mathbf{u}^1, \quad \mathbf{m}^\delta(\cdot,0) = \mathbf{m}^0, \quad |\mathbf{m}^0| = 1 \quad \text{in } \Omega$$

$$\mathbf{u}^\delta = 0, \qquad \mathbf{m}^\delta_\nu = 0 \qquad \text{on} \quad \Sigma = \partial\Omega \times [0,T]$$

Comparison: 1-D & 3-D Results

1-D

- a priori estimates;
- fixed point theorem

$\Longrightarrow$ existence & uniqueness

3-D

- Approximated penalty problem
- Faedo-Galerkin approximation
- Local Existence
- Prolongation

$\Longrightarrow$ global existence . . . but no uniqueness

The evolution problem 1-dim

Results: Singular Case(⋆)

$$\begin{cases} u_t(t) - \int_0^t G(t-\tau)u_{xx}(\tau)d\tau - u_1 - \int_0^t \frac{\lambda}{2}(\Lambda(\mathbf{m})\cdot\mathbf{m})_x d\tau = \int_0^t f(\tau)d\tau \\ \mathbf{m}_t + \mathbf{m}\frac{|\mathbf{m}|^2-1}{\delta} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0, \end{cases}$$

initial

$$u(\cdot,0) = u_0, \;\; u_t(\cdot,0) = u_1, \quad \mathbf{m}(\cdot,0) = \mathbf{m}_0, \quad |\mathbf{m}_0| = 1 \quad \text{in } \Omega$$

and boundary conditions

$$u = 0, \qquad \frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on} \quad \Sigma = \partial\Omega \times (0,T)$$

(⋆) **S. C., M. Chipot, V. Valente, G. Vergara Caffarelli, NONRWA 2017**

Theorem (1-D. Singular pb.)

- $\forall T > 0, \quad \mathcal{Q} := \Omega \times (0, T)$

$\Longrightarrow$ $\exists$ a weak solution $(u, \mathbf{m})$ to the singular problem such that:

- $u \in L^\infty(0, T; H_0^1(\Omega))$
- $u_t \in L^\infty(0, T; L^2(\Omega))$;
- $\mathbf{m} \in L^\infty(0, T; H^1(\Omega))$
- $\mathbf{m}_t \in L^2(\mathcal{Q})$.

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