Geometry and Combinatorics of Semiregular Polytopes

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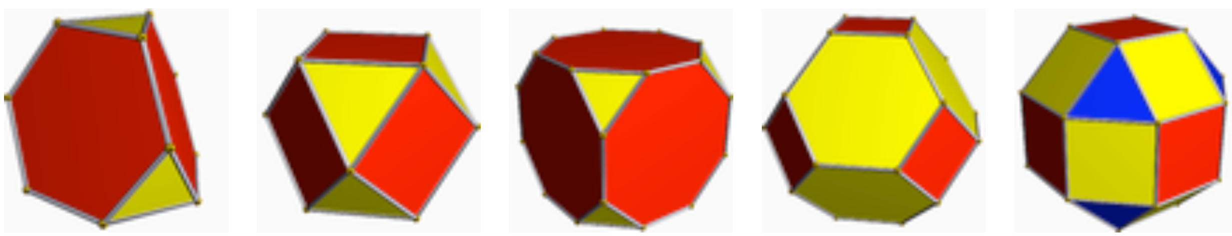
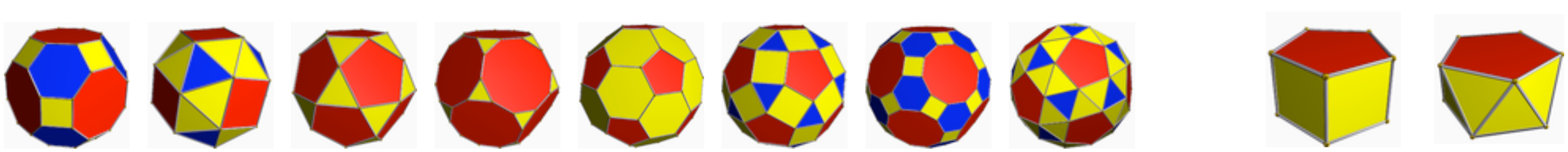
Joint work with Barry Monson.

Semiregular Convex Polyhedra

Facets are regular convex polygons. Vertex-transitive symmetry group.

- Platonic solids, Archimedean solids, prisms, antiprisms





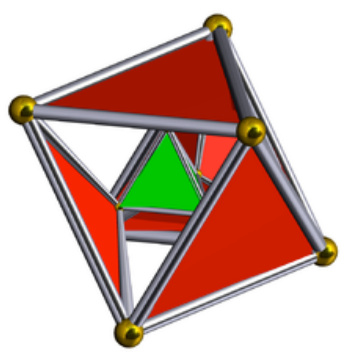
Semiregular Convex n-Polytopes, $n \geq 4$

Facets are regular $(n-1)$-polytopes. Vertex-transitive symmetry group.

Case $n = 4$: facets are Platonic solids, different shapes.

- **Three polytopes for $n = 4$, and one each for $n = 5, 6, 7, 8$, in addition to the regular polytopes.**

- Case $n = 4$: $t_1\{3, 3, 3\}$, snub 24-cell, and $t_1\{3, 3, 5\}$.



Schlegel diagram for $t_1\{3, 3, 3\}$

Facets octahedra, tetrahedra

- Case $n = 5$: half-5-cube

  (facets are 4-crosspolytopes and 4-simplices).

- Case $n = 6, 7, 8$: Gosset polytopes 2_{21}, 3_{21}, 4_{21} related to the Coxeter groups E_6, E_7 and E_8

  (via Wythoff's construction, with first node of Coxeter diagram ringed)

symbol	facet types	vert.	vertex-fig.
2_{21}	5-simpl (72), 5-crossp (27)	27	half-5-cubes
3_{21}	6-simpl (567), 6-crossp (126)	56	2_{21}
4_{21}	7-simpl (17280), 7-crossp (2160)	240	3_{21}

4_{21} is the E_8 root polytope (convex hull of root system E_8).

Semiregular polytopes are uniform polytopes.

Abstract Polytopes P of rank n

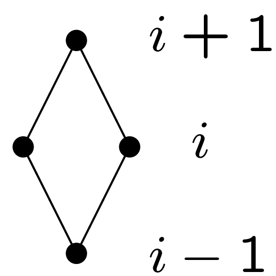
Ranked partially ordered sets, elements called faces.

(I1) Smallest and largest face (of ranks -1, n)

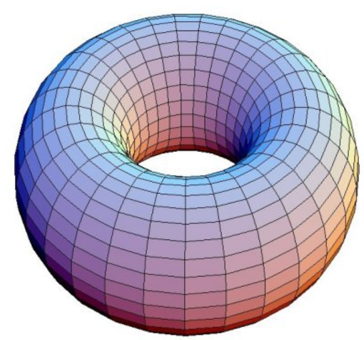
(I2) All flags contains exactly n+2 faces

(I3) P connected

(I4) Intervals of rank 1 are diamonds:



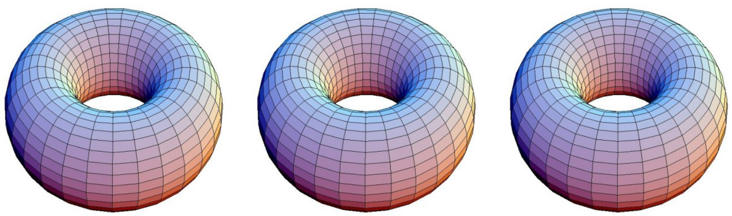
Rank 3: maps on closed surfaces



Rank $n \geq 4$: How about polytopes of rank 4?

Local picture for a 4-polytope of type $\{4, 4, 3\}$

Torus maps $\{4, 4\}_{(s,0)}$ as facets & Cubes $\{4, 3\}$ as vertex-figures



2 tori meeting at each 2-face

3 tori surround each edge

6 tori surround each vertex

P regular iff Γ(P) flag transitive.

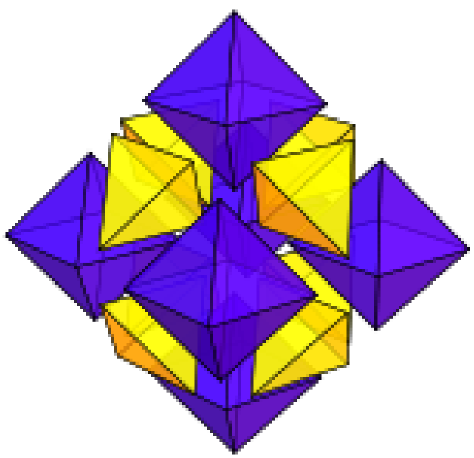
P semi-regular iff facets regular and Γ(P) vertex-transitive.

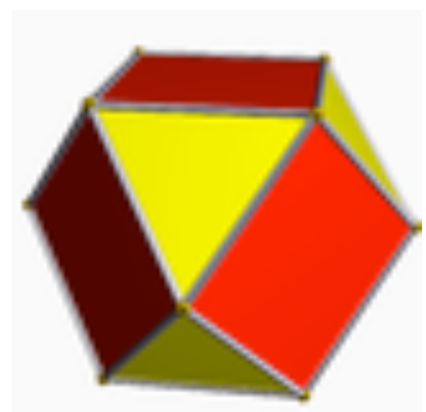
Semiregular abstract polytopes

- Facets *regular* polytopes, and automorphism group vertex-transitive.
- $n = 3$: Every vertex-transitive abstract polyhedron is semiregular.

  (Small genera: Nedela, Karabas, Pellicer, Weiss, ...)

- $n = 4$: semiregular tessellation T of $\mathbb{E}^3$ by tetrahedra and octahedra. *Alternating!* Cuboctahedral vertex-figures





Alternating semiregular polytopes S

- Facets of S all regular, group $\Gamma(S)$ vertex-transitive.

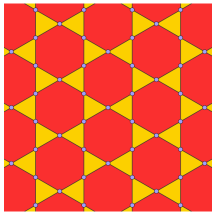
- Two kinds of regular facets P and Q *alternate* around each face of S of co-rank 2.

  Facets P and Q must be compatible: their own facets must be isomorphic, to some polytope K.

- Tomotope S: two hemi-octahedra P and two tetrahedra Q alternate around an edge of S

  2 is called the interlacing number of P and Q in S.

- $n = 2$: P triangle, Q hexagon

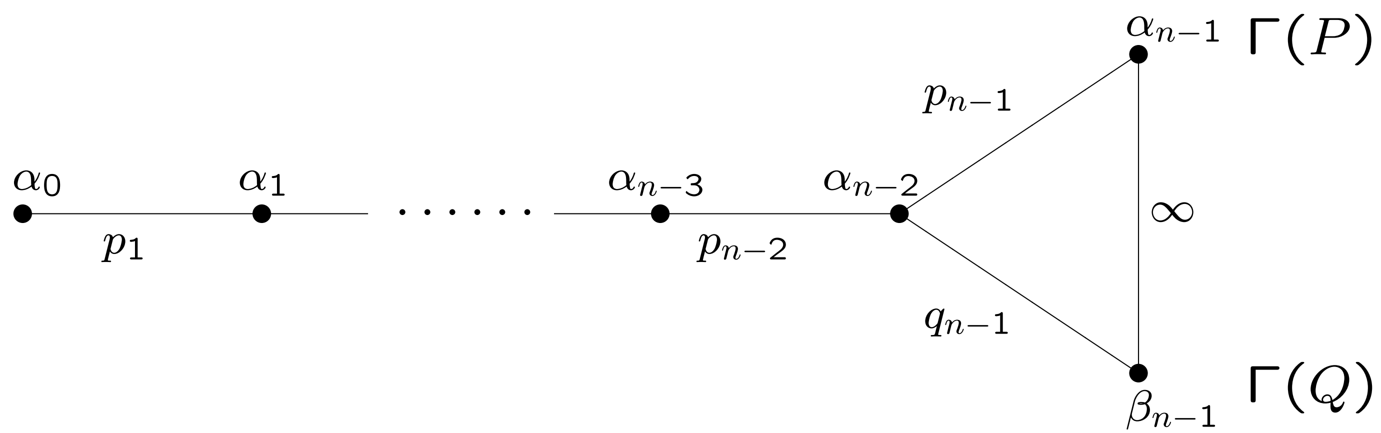


Universal alternating semiregular polytopes

Input: any two regular n-polytopes P and Q with isomorphic facets K.

- The universal $U(P,Q)$ exists for all compatible P and Q! It is semiregular and alternating, with infinitely many copies of P and Q appearing alternately at each face of co-rank 2).

- $\Gamma := \Gamma(P) *_{\Gamma(K)} \Gamma(Q) = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$



Questions

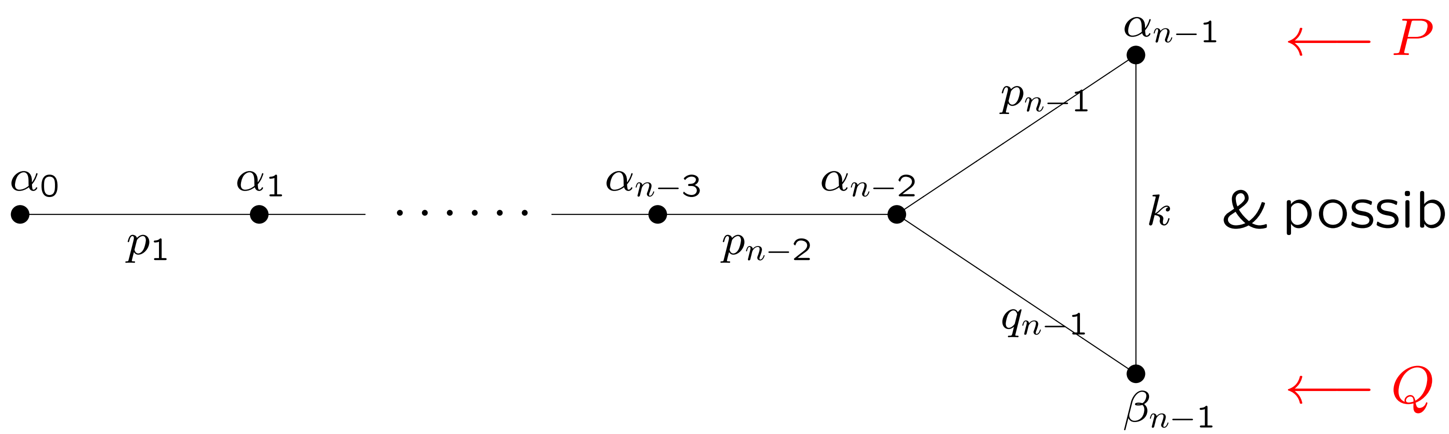
- How about finite alternating examples?

- Interlacing number k: number of facets of each kind around a ridge (corank-2 face).

- Assembly Problem: For any P and Q and any preassigned interlacing number k, does there exist an alternating semiregular $(n+1)$-polytope S with facets P and Q?

Tail-triangle group $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$,

with intersection property



& possibly more rels.

Rank $n+1$ polytope S

- *j-faces for $j \leq n-2$*

  (right) cosets of $\Gamma_j := \langle \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}, \beta_{n-1} \rangle$

- *$(n-1)$-faces*

  cosets of $\Gamma_{n-1} := \langle \alpha_0, \ldots, \alpha_{n-2} \rangle$

- n-*faces*

  cosets of Γ_n, with Γ_n either given by

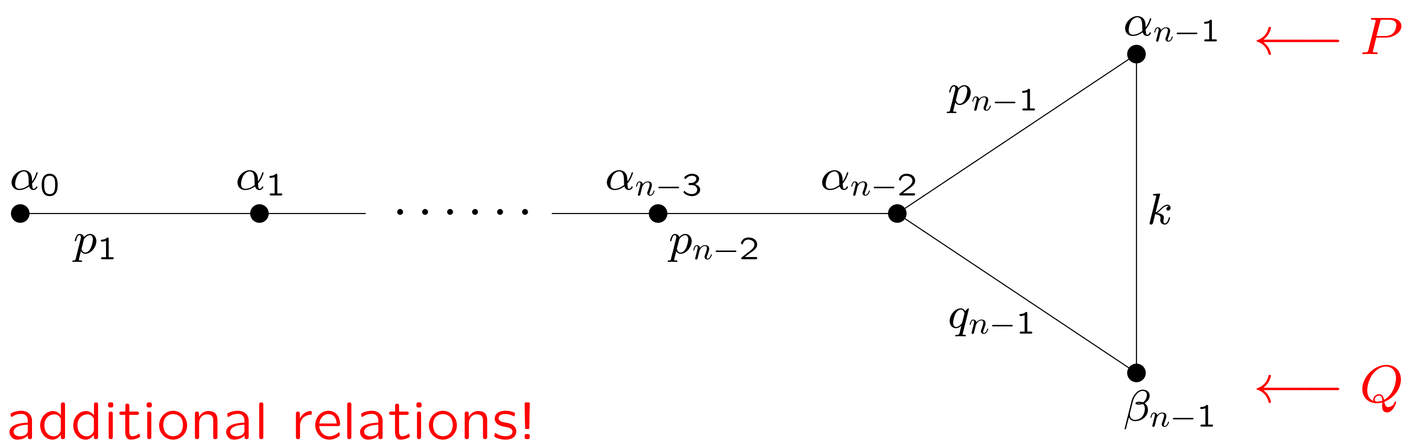
  $$\Gamma_n^P := \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \quad \text{or} \quad \Gamma_n^Q := \langle \alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle.$$

- partial order: $\Gamma_j \nu < \Gamma_k \mu$ iff $j < k$ and $\Gamma_j \nu \cap \Gamma_k \mu \neq \emptyset$

Universal $U^k(P, Q)$

Now: finite interlacing number k

Tail-triangle group $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$



No additional relations!

If any alternating semiregular polytope S with interlacing number k exists, then $U^k(P, Q)$ also exists and covers S.

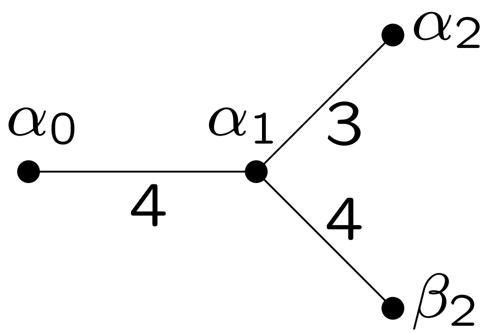
Cubes/Hemicubes

Odd interlacing number k always possible? NO!

- $P = \{4,3\}_3$ (hemicube), $Q = \{4,4\}_{(2,0)}$ (2×2 square torus). Cannot be assembled for any odd k.

Now $k = 2$.

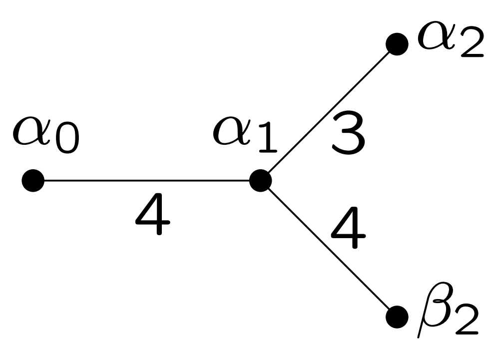
- $P = \{4,3\}_3$ (hemicube), $Q = \{4,4\}_{(s,0)}$ ($s \times s$ square torus), s odd. Cannot be assembled with $k = 2$.



& $(\alpha_0\alpha_1\beta_2)^3 = (\alpha_0\alpha_1\beta_2\alpha_1)^s = 1$

- $P = \{4,3\}_3$ (hemicube), $Q = \{4,4\}_{(s,0)}$, $s \geq 4$ even. Can be assembled with $k = 2$! Now $U^2(P,Q)$ is finite, with a group of order $24s^3$.

- $P = \{4,3\}$ (cube), $Q = \{4,4\}_{(s,0)}$, any $s \geq 2$. Can be assembled with $k = 2$!



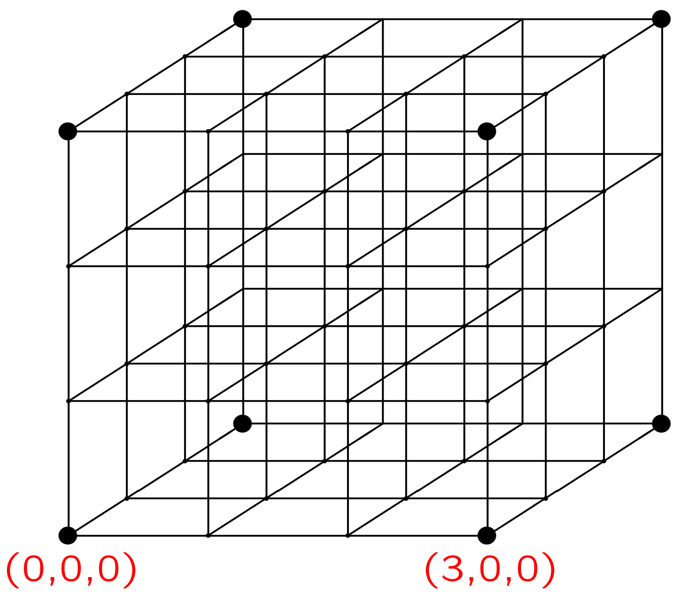
& $(\alpha_0\alpha_1\beta_2\alpha_1)^s = 1$

$S = U^2(P,Q)$ and $\Gamma(S) = W_s \ltimes D_3$, where W_s is a Coxeter group with a hexagonal diagram with all branches marked s.

S is finite iff $s = 2$, with $\Gamma(S) = C_2^6 \ltimes D_3$ if $s = 2$.

- $P = \{4,3,\ldots,3,3\}_n$ (n-hemicube) and $Q = \{4,3\ldots,3,4\}_{(s,0,\ldots,0)}$ ($s \times s \times \ldots \times s$ cubical n-toroid).

Cannot be assembled with $k = 2$, for any $n \geq 3$ and s odd!



$3 \times 3 \times 3$ cubical toroid $\{4,3,4\}_{(3,0,0)}$

- How about hemi-octahedra and tetrahedra as facets?

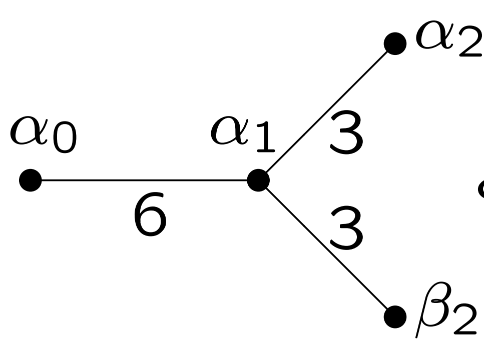
Tomotope: 4 vertices, 12 edges, 16 triangles, 4 tetrahedra & 4 hemi-octahedra.

Vertex-figures hemi-cuboctahedra. Group order 96.

(Monson, Pellicer, Williams)

Hexagonal torus map

- $P = \{6,3\}_{(1,1)}$, and Q either $\{6,3\}$ or any map $\{6,3\}_{(b,c)}$



& $(\alpha_0\alpha_1\alpha_2)^2 = 1$ & one rel.

Assembly with $k = 2$ only possible when $Q = \{6,3\}_{(2,2)}$ or $Q = P$ (in this case Γ has order 576 or 144).

$P = \{6,3\}_{(1,1)}$ and $Q = \{6,3\}_{(2,0)}$ cannot be assembled for $k = 4$ either! Group collapses!

Some conjectures

- Conj. 1: If $U^k(P,Q)$ exists and $k\,|\,m$, then $U^m(P,Q)$ exists.

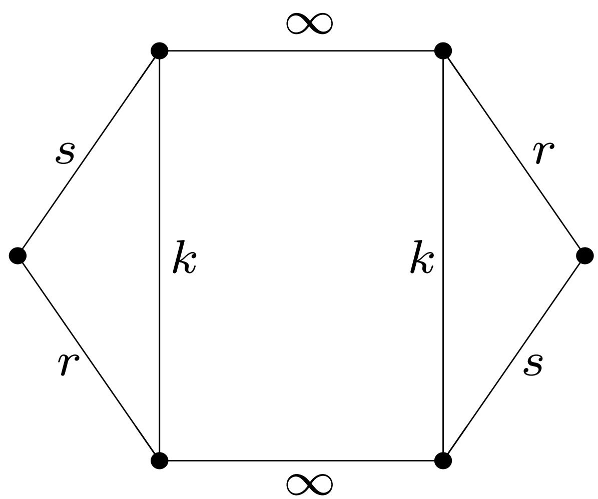
- Conj. 2: Given compatible P,Q there are infinitely many integers $k \geq 2$ such that $U^k(P,Q)$ exist.

(Recall that $U^\infty(P,Q)$ exists! Maybe for k large enough.)

- Conj. 3: For any even $k \geq 2$, there exist compatible P,Q such that $U^k(P,Q)$ does not exist.

Universal $U^k(P,Q)$ when $P = \{4,4\}_{(r,0)}$, $Q = \{4,4\}_{(s,0)}$ ($r,s,k \geq 2$)

$U^k(P,Q)$ has group $W(r,s,k) \ltimes C_2$, where $W(r,s,k)$ is the Coxeter group with diagram



$U^k(P,Q)$ always infinite, as are its vertex-figures.

Some constructions

- Small examples with $k = 2$ exist for many choices of P, Q!

If $\Gamma(P) = N_P \ltimes \Gamma(K)$ and $\Gamma(Q) = N_Q \ltimes \Gamma(K)$, then

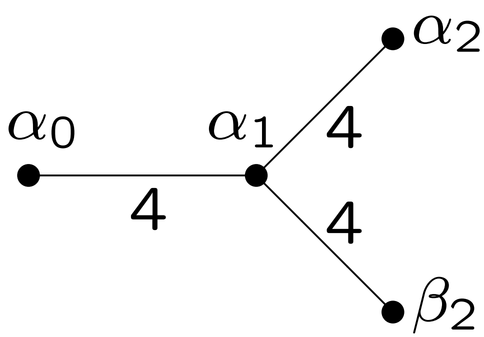
$$\Gamma := (N_P \times N_Q) \ltimes \Gamma(K)$$

is a tail-triangle C-group giving an alternating semiregular polytope S with facets P, Q and $k = 2$.

Smallness : $\mathrm{f}_0(\mathrm{S}) = \mathrm{f}_0(\mathrm{P})\mathrm{f}_0(\mathrm{Q})/\mathrm{f}_0(\mathrm{K}),\ \ \mathrm{f}_\mathrm{n}(\mathrm{S}) = \mathrm{f}_{\mathrm{n}-1}(\mathrm{P}) + \mathrm{f}_{\mathrm{n}-1}(\mathrm{Q})$.

Examples: P, Q taken from $\{4,4\}_{(2t,0)}$, $\{4,4\}_{(2t,2t)}$ or $\{3,6\}_{(3t,0)}$, for $t \geq 1$.

$P = \{4,4\}_{(2s,0)}$, $Q = \{4,4\}_{(2t,0)}$



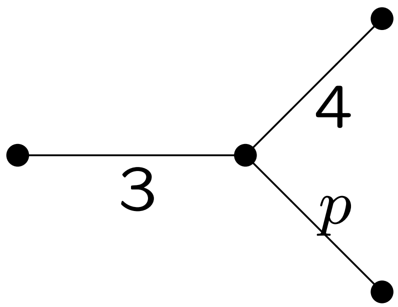
& $(\alpha_0\alpha_1\alpha_2\alpha_1)^{2s} = (\alpha_0\alpha_1\beta_2\alpha_1)^{2t} = 1$

$\Gamma = (D_s \times D_s \times D_t \times D_t) \ltimes D_4$,

$\#(\text{vertices of S}) = 4s^2t^2$, $\quad \#(\text{facets of S}) = 4(s^2 + t^2)$

• Many finite examples by modular reduction from crystallographic Coxeter groups!

p prime, $p \neq 2$:



$O(4, p, \pm 1)$, $O_1(4, p, \pm 1)$

$P = \{3, 4\}$ (octahedron), Q a certain map of type $\{3, p\}$ with group $\mathrm{PSL}(\mathbb{Z}_p) \ltimes C_2$ or $\mathrm{PGL}(\mathbb{Z}_p) \ltimes C_2$

$p = 3, 5, 7$: Q is the tetrahedron $\{3, 3\}$, icosahedron $\{3, 5\}$, or Klein map $\{3, 7\}_8$, resp.

Future work by Kadin (Finn) Prideaux!

Abstract

Geometry and Combinatorics of Semiregular Polytopes

Traditionally, a polyhedron or polytope is semiregular if its facets are regular and its symmetry group is transitive on vertices. We briefly review the semiregular convex polytopes, and then discuss semiregular abstract polytopes, which have abstract regular facets, still with combinatorial automorphism group transitive on vertices. Our focus is on alternating semiregular polytopes, with two kinds of regular facets occurring in an alternating fashion. The cuboctahedron is a familiar example in rank 3. We then describe recent progress on the assembly problem for alternating semiregular polytopes: which pairs of regular n-polytopes can occur as facets

of a semiregular (n+1)-polytope. If time permits, we brief discuss semiregularity in the context of skeletal polyhedra in 3-space. Most work is joint with Barry Monson.