A Neumann p-Laplacian problem on metric spaces

Antonella Nastasi

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From the Euclidean to the metric setting

Let $\Omega \subset \mathbb{R}^N$. The Neumann boundary value problem driven by a p-Laplacian operator is

$$\begin{cases} -\Delta_p u = g & \text{in } \Omega, \\ -|\nabla u|^{p-2}\partial_\eta u = f & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $1 < p < \infty$, g is a continuous function and $\partial_\eta u$ is the directional derivative of u in the direction of the outer normal to $\partial\Omega$.

From the Euclidean to the metric setting

The weak formulation of the problem is to find $u \in W^{1,p}(\Omega)$ such that

$$\int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x)\nabla\varphi(x)dx - \int_{\partial\Omega}\varphi(x)f(x)d\mathcal{H}^{n-1}(x) = \int_{\Omega} g(u(x))\varphi(x)dx,$$

for all $\varphi \in W^{1,p}(\Omega)$.

Thus, solving (1) reduces to look for critical points of the p-energy functional

$$J(u) = \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} G(u)dx + \int_{\partial\Omega} ufd\mathcal{H}^{n-1},$$

where G is a primitive of g.

Applications

- Calculus on Riemannian manifolds.
- Subelliptic operators associated with vector fields.
- Potential theory on graphs.
- Weighted Sobolev spaces.

Some references

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The problem

Let X be a complete metric space equipped with a doubling measure supporting a $(1, p)$-Poincaré inequality $(1 < p < +\infty)$.

Given a Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G, we consider the following functional

$$J(u) = \int_\Omega g_u^p d\mu - \int_\Omega G(u) d\mu + \int_{\partial\Omega} Tu\, f\, dP_\Omega \quad \text{for all } u \in N^{1,p}(\Omega). \quad (2)$$

where

- Ω is a bounded domain (non empty, connected open set) in X with $X \setminus \Omega$ of positive measure such that Ω is of finite perimeter with perimeter measure P_Ω;

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where

- $G : \Omega \to \mathbb{R}$ is defined as

$$G(u) = c - |u|^\gamma \text{ for all } u \in N^{1,p}(\Omega), \qquad (3)$$

for some $c > 0$ and $1 < \gamma < p^* = \frac{ps}{s-p}$ if $p < s$ and $1 < \gamma < +\infty$ otherwise;

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where

- $f : \partial\Omega \to \mathbb{R}$ is a bounded P_Ω-measurable function with $\int_{\partial\Omega} f dP_\Omega = 0$.

Solution to the Neumann boundary value problem

Definition

A function $u_0 \in N^{1,p}_*(\Omega)$ is a p-harmonic solution to the Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G if

$$J(u_0) = \int_\Omega g^p_{u_0} d\mu - \int_\Omega G(u_0) d\mu + \int_{\partial\Omega} Tu_0 f dP_\Omega \leq \int_\Omega g^p_v d\mu - \int_\Omega G(v) d\mu + \int_{\partial\Omega} Tv f dP_\Omega = J(v)$$

for every $v \in N^{1,p}_*(\Omega)$, where g_{u_0}, g_v are the minimal p-weak upper gradients of u_0 and v in Ω, respectively, and Tu_0 and Tv are the traces of u_0 and v on $\partial\Omega$, respectively.

Overview of the results obtained

- existence of a solution and a weaker uniqueness property;
- minimizers of the Neumann p-Laplacian problem satisfy a De Giorgi type inequality and consequently we give boundedness properties for them;
- minimizers of the Neumann p-Laplacian problem with zero boundary data are in the De Giorgi class. This permits us to prove some further regularity results.

Doubling measure

Let (X, d, μ) be a metric measure space, where μ is a Borel regular measure. Let $B(x, \rho) \subset X$ be a ball with the center $x \in X$ and the radius $\rho > 0$.

Definition ((Björn, Björn (2011)), Section 3.1)

A measure μ on X is said to be doubling if there exists a constant K, called the doubling constant, such that

$$0 < \mu(B(x, 2\rho)) \leq K\mu(B(x, \rho)) < +\infty,$$

for all $x \in X$ and $\rho > 0$.

$(1, p)$-Poincaré inequality

For a measurable set $S \subset X$ of finite positive measure and for a measurable function $u : S \to \mathbb{R}$, we denote

$$u_S = \frac{1}{\mu(S)} \int_S u d\mu.$$

Definition ((Björn, Björn (2011)), Definition 4.1)

Let $p \in [1, +\infty[$. A metric measure space X supports a $(1, p)$-Poincaré inequality if there exist $K > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq Kr \left(\frac{1}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r)} g_u^p \, d\mu \right)^{\frac{1}{p}}$$

for all balls $B(x, r) \subset X$ and for all $u \in L^1_{loc}(X)$.

Upper gradient

Definition ((Björn, Björn (2011)), Definition 1.13)

A non negative Borel measurable function g is said to be an upper gradient of function $u : X \to [-\infty, +\infty]$ if, for all compact rectifiable arc length parametrized paths γ connecting x and y, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds \tag{4}$$

whenever $u(x)$ and $u(y)$ are both finite and $\int_\gamma g \, ds = +\infty$ otherwise.

p-weak upper gradient

Definition ((Björn, Björn (2011)), Definition 1.33)

Let $p \in [1, +\infty[$. Let Γ be a family of paths in X. We say that

$$\inf_{\phi} \int_X \phi^p d\mu$$

is the p-modulus of Γ, where the infimum is taken among all non negative Borel measurable functions ϕ satisfying $\int_\gamma \phi \, ds \geq 1$, for all rectifiable paths $\gamma \in \Gamma$.

Definition ((Björn, Björn (2011)), Definition 1.32)

If (4) is satisfied for p-almost all paths γ in X, that is the set of non constant paths that do not satisfy (4) is of zero p-modulus, then g is said a p-weak upper gradient of u.

minimal p-weak upper gradient

The family of weak upper gradients satisfy the result contained in the following theorem concerning the existence of a minimal element.

Theorem ((Björn, Björn (2011)), Theorem 2.5)

Let $p \in]1, +\infty[$. Suppose that $u \in L^p(X)$ has an $L^p(X)$ integrable p-weak upper gradient. Then there exists a p-weak upper gradient, denoted with g_u, such that $g_u \leq g$ μ-a.e. in X, for each p-weak upper gradient g of u. This g_u is called the minimal p-weak upper gradient of u.

We note that g_u is μ-a.e. uniquely determinated by u.

The Newtonian space

Let X be a complete metric space equipped with a doubling measure supporting a $(1, p)$-Poincaré inequality.

Definition

The Newtonian space $N^{1,p}(X)$ is defined by

$$N^{1,p}(X) = V^{1,p}(X) \cap L^p(X), \quad p \in [1, +\infty],$$

where $V^{1,p}(X) = \{u : u \text{ is measurable and } g_u \in L^p(X)\}$. We consider $N^{1,p}(X)$ equipped with the norm

$$\|u\|_{N^{1,p}(X)} = \|g_u\|_{L^p(X)} + \|u\|_{L^p(X)}.$$

We denote with $N^{1,p}_*(X) = \{u \in N^{1,p}(X) : \int_X u\, dx = 0\}$.

The Newtonian space $N^{1,p}(X)$ is a complete normed vector space, which generalizes the Sobolev space $W^{1,p}(\Omega)$ to a metric setting.

The perimeter

Definition (see (Miranda (2003)))

A Borel set $E \subset X$ is said to be of finite perimeter if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $N^{1,1}(X)$ such that $u_n \to \chi_E$ in $L^1(X)$ and

$$\liminf_{n\to+\infty} \int_X g_{u_n} d\mu < \infty.$$

The perimeter $P_E(X)$ of E is the infimum of the above limit among all sequences $\{u_n\}$ as above. For an open set $U \subset X$, the perimeter of E in U is

$$P_E(U) = \inf\left\{\liminf_{n\to+\infty} \int_X g_{u_n} d\mu : \{u_n\}_{n\in\mathbb{N}} \subset N^{1,1}(U), u_n \to \chi_{E\cap U} \text{ in } L^1(U)\right\}.$$

We note that E is a set of finite perimeter iff χ_E is a BV(U) function (Miranda (2003), Definition 4.1).

Hypotheses set on Ω

(H_1) There exists a constant $K \geq 1$ such that for all $y \in \Omega$ and $0 < \rho \leq \mathrm{diam}\,(\Omega)$, we have

$$\mu(B(y,\rho) \cap \Omega) \geq \frac{1}{K}\mu(B(y,\rho)).$$

(H_2) (Ahlfors codimension 1 regularity of P_Ω) For all $y \in \partial\Omega$ we have that

$$\frac{1}{K\rho}\mu(B(y,\rho)) \leq P_\Omega(B(y,\rho)) \leq \frac{K}{\rho}\mu(B(y,\rho)),$$

where K and ρ are as in (H_1).

(H_3) $(\Omega, d_{|\Omega}, \mu_{|\Omega})$ admits a $(1,p)$-Poincaré inequality with $\lambda = 1$, where $p \in]1, +\infty[$.

Trace operator

Definition (Lahti (2015), Definition 4.1)

Let $\Omega \subset X$ be an open set and let u be a μ-measurable function on Ω. A function $Tu : \partial\Omega \to \mathbb{R}$ is the trace of u if for $\mathcal{H}$-almost every $y \in \partial\Omega$ we have

$$\lim_{\rho \to 0^+} \frac{1}{\mu(\Omega \cap B(y,\rho))} \int_{\Omega \cap B(y,\rho)} |u - Tu(y)| d\mu = 0.$$

Existence of a solution and a weaker uniqueness result

The existence of a nontrivial solution to the Neumann boundary value problem with non zero boundary data f and reaction term G is an immediate consequence of the following theorem which shows that J has a minimizer.

Theorem

- *J has a minimizer in $N^{1,p}_*(\Omega)$.*
- *If $u_1, u_2 \in N^{1,p}_*(\Omega)$ are two minimizers of J, then $g_{u_1} = g_{u_2}$ a.e. in Ω.*

Boundedness property

We show that minimizers are locally bounded near the boundary under appropriate hypothesis on the boundary data f.
We assume that $f \in L^\infty(\partial\Omega)$. Our aim is to prove that, under this assumption, we get that $u \in L^\infty(\Omega_R)$ and $Tu \in L^\infty(\partial\Omega_R)$ where

$$\Omega_R = \left\{ y \in \Omega : d(y, \partial\Omega) < \frac{R}{2} \right\} \tag{5}$$

for an appropriate $R > 0$, that is u is bounded near the boundary.

De Giorgi type inequality

Lemma

Let $u \in N^{1,p}_*(\Omega)$ be a minimizer of J and $f \in L^\infty(\partial\Omega)$. If $y \in \partial\Omega$, $0 < \rho < R < \frac{\mathrm{diam}(\Omega)}{10}$ and $\alpha \in \mathbb{R}$, then there is $K \geq 1$ such that the following De Giorgi type inequality

$$\int_{\Omega\cap B(y,\rho)} g^p_{(u-\alpha)_+}\,d\mu \leq \frac{K}{(R-\rho)^p}\int_{\Omega\cap B(y,R)} (u-\alpha)^p_+\,d\mu + K\int_{\partial\Omega\cap B(y,R)} |f|(u-\alpha)^p_+\,dP_\Omega \tag{6}$$

is satisfied.

The proof

We define

$$\tau_{\rho,R}(x) = \tau(x) = \left(1 - \frac{d(x, B(y,\rho))}{R-\rho}\right)_+$$

and

$$S_{\alpha,r} = \{x \in B(y,r) \cap \Omega : u(x) > \alpha\} \cup \{x \in B(y,r) \cap \partial\Omega : u(x) > \alpha\}.$$

We consider

$$w = u - \tau(u-\alpha)_+ = \begin{cases} (1-\tau)(u-\alpha) + \alpha & \text{in } S_{\alpha,R} \\ u & \text{otherwise.} \end{cases} \tag{7}$$

The proof

We observe that, from the definition of w, we have $|w| \leq |u|$. Using Leibniz rule,

$$g_w \leq \begin{cases} (1-\tau)g_u + \dfrac{u-\alpha}{R-\rho}\chi_{B(y,R)\setminus B(y,\rho)} & \text{in } S_{\alpha,R} \\ g_u & \text{otherwise.} \end{cases} \tag{8}$$

By (8) we deduce that

$$g_w^p \leq 2^p \left(g_u^p (1 - \chi_{S_{\alpha,\rho}}) + \frac{(u-\alpha)^p}{(R-\rho)^p} \right) \quad \text{in } S_{\alpha,R}. \tag{9}$$

The proof

Since u is a minimizer of J, then

$$\begin{aligned} J(u) &= \int_{\Omega\cap B(y,R)} g_u^p d\mu - \int_{\Omega\cap B(y,R)} (c - |u|^\gamma) d\mu + \int_{\partial\Omega\cap B(y,R)} ufdP_\Omega \\ &\leq \int_{\Omega\cap B(y,R)} g_w^p d\mu - \int_{\Omega\cap B(y,R)} (c - |w|^\gamma) d\mu + \int_{\partial\Omega\cap B(y,R)} wfdP_\Omega \\ &= J(w). \end{aligned} \tag{10}$$

The proof

By adding
$-\int_{\Omega\cap B(y,R)\setminus S_{\alpha,R}} g_u^p d\mu + \int_{\Omega\cap B(y,R)}(c-|u|^\gamma)d\mu - \int_{\partial\Omega\cap B(y,R)} ufdP_\Omega$
to both sides of (10), we get

$$\begin{aligned}\int_{S_{\alpha,R}} g_u^p d\mu &\leq \int_{S_{\alpha,R}} g_w^p d\mu - \int_{\Omega\cap B(y,R)}(c-|w|^\gamma-(c-|u|^\gamma))d\mu \\ &\quad - \int_{\partial\Omega\cap S_{\alpha,R}} \tau(u-\alpha)fdP_\Omega \\ &\leq \int_{S_{\alpha,R}} g_w^p d\mu - \int_{\Omega\cap B(y,R)}(|u|^\gamma-|w|^\gamma)d\mu \\ &\quad - \int_{\partial\Omega\cap S_{\alpha,R}} \tau(u-\alpha)fdP_\Omega \\ &\leq \int_{S_{\alpha,R}} g_w^p d\mu - \int_{\partial\Omega\cap S_{\alpha,R}} \tau(u-\alpha)fdP_\Omega \quad \text{(by (7))}. \qquad (11)\end{aligned}$$

The proof

Using (9) and (11), we obtain

$$\int_{S_{\alpha,\rho}} g_u^p d\mu \leq 2^p \int_{S_{\alpha,R} \setminus S_{\alpha,\rho}} g_u^p d\mu + \frac{2^p}{(R-\rho)^p} \int_{S_{\alpha,R}} (u-\alpha)^p d\mu - \int_{\partial\Omega \cap S_{\alpha,R}} \tau(u-\alpha) f dP_\Omega.$$

The proof

Now, we add $2^p \int_{S_{\alpha,\rho}} g_u^p d\mu$ to both sides of the inequality, then we divide all by $1+2^p$ and we obtain

$$\int_{S_{\alpha,\rho}} g_u^p d\mu \leq \frac{2^p}{1+2^p}\int_{S_{\alpha,R}} g_u^p d\mu + \frac{2^p}{(1+2^p)(R-\rho)^p}\int_{S_{\alpha,R}} (u-\alpha)^p d\mu - \frac{1}{(1+2^p)}\int_{\partial\Omega\cap S_{\alpha,R}} \tau(u-\alpha) f dP_\Omega. \tag{12}$$

The proof

At this point we can use (12) and a lemma by Giusti (Giusti, Direct Methods in the Calculus of Variations. World Scientific Publishing, River Edge (2003), Lemma 6.1) to get

$$\int_{S_{\alpha,\rho}} g_u^p d\mu \leq \frac{K}{(R-\rho)^p} \int_{S_{\alpha,R}} (u-\alpha)^p d\mu + K \int_{\partial\Omega \cap S_{\alpha,R}} \tau(u-\alpha)|f| dP_\Omega.$$

That completes the proof.

A boundedness result

Theorem

Let $0 < R < \frac{\mathrm{diam}(\Omega)}{4}$ *and* $\Omega_R = \left\{ y \in \Omega : d(y, \partial\Omega) < \frac{R}{2} \right\}$.

If $u \in N_*^{1,p}(\Omega)$ *is a minimizer of* J *and* $f \in L^\infty(\partial\Omega)$, *then*

$$u \in L^\infty(\Omega_R) \text{ and } Tu \in L^\infty(\partial\Omega_R).$$

Sketch of the proof

Proceeding as in the proof of [Malý and Shanmugalingam (2018), Theorem 5.2], we can find $d \geq 0$ such that

$$\int_{\Omega\cap B\left(x,\frac{R}{2}\right)} (u-d)_+^p \, d\mu = 0, \quad \text{for all } x \in \partial\Omega.$$

This implies that $u \leq d$ $\mu-$a.e. in $\Omega \cap B\left(x, \frac{R}{2}\right)$. Consequently, $u \leq d$ $\mu-$a.e. in Ω_R.

Sketch of the proof

In order to deduce that u is also $\mu-$a.e. lower bounded, we observe that if u is a minimizer for J, then $-u$ is a minimizer for J_-, where J_- is defined as

$$J_-(u) = \int_\Omega g_u \, d\mu - \int_\Omega (c - |u|^\gamma) d\mu - \int_{\partial\Omega} ufdP_\Omega.$$

In fact, u minimizer for J means $J(u) \leq J(v)$ for all $v \in N_*^{1,p}(\Omega)$.

Sketch of the proof

We have that

$$J_{-}(-u) = J(u) \leq J(v) = J_{-}(-v) \quad \text{for all } v \in N_{*}^{1,p}(\Omega),$$

which means that $-u$ is a minimizer of J_{-}. This ensures that $-u$ is $\mu-$a.e. upper bounded in Ω_R and so u is $\mu-$a.e. lower bounded in Ω_R. We conclude that $u \in L^{\infty}(\Omega_R)$. In a similar way, we have that $Tu \in L^{\infty}(\partial\Omega_R)$.

Other results and forthcoming research

- Neumann p-Laplacian problem with zero boundary data;
- Extending the results to the (p, q)- Laplacian problem in the metric setting.

  [Joint paper with Cintia Pacchiano Camacho]
  Given a Dirichlet (p, q)-boundary value problem, we associate the following functional

  $$J(u) = \int_\Omega g_u^p d\mu + \int_\Omega g_u^q d\mu \quad \text{for all } u \in N^{1,p}_{loc}(\Omega). \tag{13}$$

  in the setting of a non empty open set Ω of a metric measure space (X, d, μ) equipped with a doubling Borel regular measure μ and supporting a weak $(1, s)$-Poincaré inequality for some s such that $1 < s < q < p < s^*$, where s^* is the critical exponent associated to s.

Other results and forthcoming research

Definition

A function $u_0 \in N^{1,p}_{loc}(\Omega)$ is a quasi-minimizer of J on Ω if there exists $C \geq 1$ such that for every bounded open subset Ω' of Ω with $\overline{\Omega'} \subset \Omega$ and for all functions $v \in N^{1,p}(\Omega')$ with $u_0 - v \in N^{1,p}_0(\Omega')$ the inequality

$$\int_{\Omega'} g_{u_0}^p d\mu + \int_{\Omega'} g_{u_0}^q d\mu \leq C \left(\int_{\Omega'} g_v^p d\mu + \int_{\Omega'} g_v^q d\mu \right)$$

holds, where g_{u_0}, g_v are the minimal p-weak upper gradients of u_0 and v in Ω, respectively.

Other results and forthcoming research

Definition

Let $L^p_{loc}(\Omega)$ be the space of all measurable functions that are p-integrable on bounded subsets of X.
The space $N^{1,p}_{loc}(\Omega)$ is defined by

$$N^{1,p}_{loc}(\Omega) = V^{1,p}_{loc}(\Omega) \cap L^p_{loc}(\Omega), \quad p \in [1, +\infty],$$

where $V^{1,p}_{loc}(\Omega) = \{u : u \text{ is measurable and } g_u \in L^p_{loc}(\Omega)\}$.

Other results and forthcoming research

Lemma

Let $u \in N^{1,p}_{loc}(\Omega)$ be a quasi minimizer of J. If $0 < \rho < R < \frac{\mathrm{diam}(\Omega)}{3}$, then there exists $c_1, c_2 \geq 0$ such that the following De Giorgi type inequality

$$\int_{S_{\alpha,\rho}} (g_u^p + g_u^q) d\mu \leq \frac{c_1}{(R-\rho)^p} \int_{S_{\alpha,R}} (u-\alpha)^p d\mu + \frac{c_2}{(R-\rho)^q} \int_{S_{\alpha,R}} (u-\alpha)^q d\mu,$$

is satisfied.

Other results and forthcoming research

De Giorgi type inequality has a key role in order to prove...

- boundedness results
- other regularity results as Hölder continuity, Harnack's inequality, strong maximum principle ...
  (to be continued)

Thanks for your attention!