The two-variable Bollobás–Riordan polynomial of a connected even delta-matroid is irreducible

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The Tutte polynomial

The Tutte polynomial of a graph G is given by

$$T(G;x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A) = |V| - k(G|A)$, the number of edges in the largest forest of $G|A$.

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Equivalently, $T(G; x, y) = 1$ if G has no edges, and for each edge e,

$$T(G; x, y) = \begin{cases} xT(G/e; x, y) & \text{if } e \text{ is a bridge,} \\ yT(G \setminus e; x, y) & \text{if } e \text{ is a loop,} \\ T(G/e; x, y) + T(G \setminus e; x, y) & \text{otherwise.} \end{cases}$$

Irreducibility of T

Theorem (Merino, de Mier, Noy (2001))

$T(G; x, y)$ is irreducible in $\mathbb{C}[x, y]$ if and only if G is 2-connected.

(This is also true for matroids.)

Key facts used in the proof

Write $T(G;x,y) = \sum_{i,j} b_{i,j} x^i y^j$. (We have $b_{i,j} \geq 0$.)

a. Brylawski's affine identities. For example,
   1. if G has at least one edge then $b_{0,0} = 0$;
   2. if G has at least two edges then $b_{1,0} = b_{0,1}$.
b. If G has at least 2 edges, then $b_{1,0} \neq 0$ if and only if G is 2-connected.
c. $T(G;x,y)$ has degree $r(E)$ in x and if G is loopless then $b_{r(E),0} = 1$ and otherwise $b_{r(E),i} = 0$.
d. $T(G;x,y)$ has degree $|E| - r(E)$ in y and if G is bridgeless then $b_{0,|E|-r(E)} = 1$ and otherwise $b_{i,|E|-r(E)} = 0$.

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Let $\sigma(A) = r(A) + g(A)$.

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We have

$$R(\mathbb{G}; x, y) = (x-1)^{g(\mathbb{G})} BR(\mathbb{G}, x, y-1, 1/\sqrt{(x-1)(y-1)}),$$

where $BR(\mathbb{G})$ is the Bollobás–Riordan polynomial of $\mathbb{G}$.

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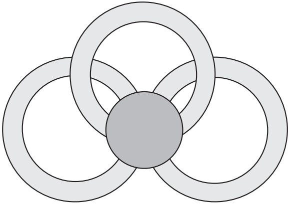
If $\mathbb{G}$ is a plane graph, then $R(\mathbb{G}) = T(G)$.

An example



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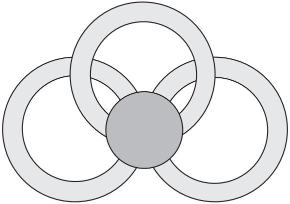

In this graph $\sigma(E) = 1$.

$$R(\mathbb{G}) = (x-1) + 3(x-1)(y-1) + (x-1)(y-1)^2 + 2(y-1) + (y-1)^2 = xy^2 + xy - x - y,$$

which is irreducible.

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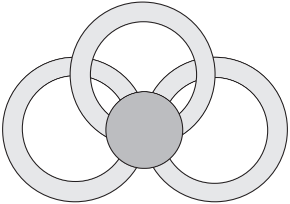

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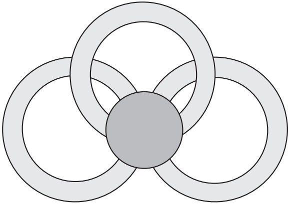

A ribbon graph is not 2-connected if its edges can be partitioned into sets *A* and *B*, so that each circuit lies in either *A* or *B* and no circuit in *A* is interlaced with a circuit in *B*.

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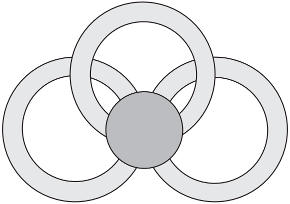

A ribbon graph is not 2-connected if its edges can be partitioned into sets *A* and *B*, so that each circuit lies in either *A* or *B* and no circuit in *A* is interlaced with a circuit in *B*.

A loop is trivial if it is not interlaced with any other circuit.

Delete–Contract

If $\mathbb{G}$ has no edges than $R(\mathbb{G}; x, y) = 1$ and for each edge e,

- If e is not a loop in either $\mathbb{G}$ or $\mathbb{G}^*$, then

$$R(\mathbb{G}; x, y) = R(\mathbb{G} \setminus e; x, y) + R(\mathbb{G}/e; x, y).$$

- If e is a loop in $\mathbb{G}$ but not in $\mathbb{G}^*$, then

$$R(\mathbb{G}; x, y) = (x - 1)R(\mathbb{G} \setminus e; x, y) + R(\mathbb{G}/e; x, y).$$

- If e is a loop in $\mathbb{G}^*$ but not in $\mathbb{G}$, then

$$R(\mathbb{G}; x, y) = R(\mathbb{G} \setminus e; x, y) + (y - 1)R(\mathbb{G}/e; x, y).$$

- If e is a loop in both $\mathbb{G}$ and $\mathbb{G}^*$, then

$$R(\mathbb{G}; x, y) = (x - 1)R(\mathbb{G} \setminus e; x, y) + (y - 1)R(\mathbb{G}/e; x, y).$$

Key facts

Write $R(\mathbb{G}; x, y) = \sum_{i,j} r_{i,j} x^i y^j$. (We no longer have $r_{i,j} \geq 0$.)

a. Brylawski's affine identities.

   Gordon (2015) showed that Brylawski's affine identities hold extremely generally.

b. *If G has at least 2 edges, then $b_{1,0} \neq 0$ if and only if G is 2-connected.*

   If $\mathbb{G}$ has at least 2 edges, then $r_{1,0} \neq 0$ if and only if $\mathbb{G}$ is 2-connected. This follows from a result of Bouchet (2001), which implies that if $\mathbb{G}$ is 2-connected then at least one of $\mathbb{G} \setminus e$ and $\mathbb{G}/e$ is 2-connected.

c. *$T(G; x, y)$ has degree $r(E)$ in x and if G is loopless then $b_{r(E),0} = 1$ and otherwise $b_{r(E),i} = 0$.*

   If $i > \sigma(E)$, then $r_{i,j} = 0$. Moreover

$$\sum_j r_{\sigma(E),j} = 1.$$

Main theorem

Theorem

If $\mathbb{G}$ is an orientable ribbon graph, then $R(\mathbb{G}; x, y)$ is irreducible if and only if $\mathbb{G}$ is 2-connected.

(This extends to even delta-matroids.)

Thanks and questions

Thank you for listening.