The use of rational approximation for linearization of models that are nonlinear in the frequency

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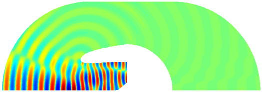
Analysis of vibrations

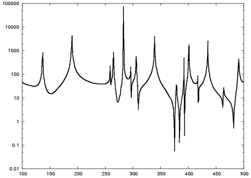
- 'Classical' analysis (frequency domain): Helmholtz equation
- Discretization: FE, BE, Trefftz

$$(K - \omega^2 M)x = f \qquad\qquad (K + \imath\omega C - \omega^2 M)x = f$$



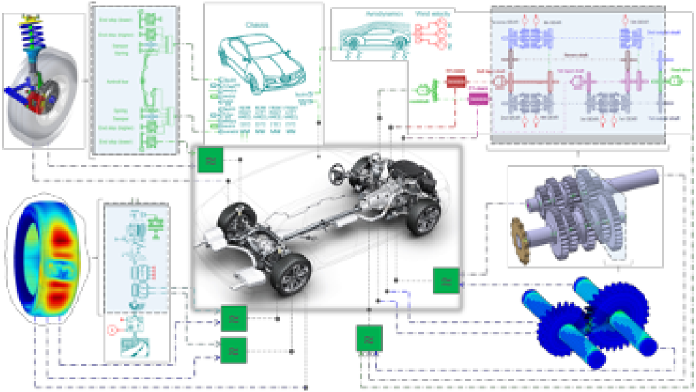
Simple ω dependency $\Longrightarrow$

- Frequency sweeping (computing x for many ω)
- Time stepping (connection between Fourier domain and time domain)
- Eigenvalue computations



Trends in the analysis of vibrations

- Nonlinear frequency dependencies
- Nonlinear time dependent models (mechatronic systems)
- Digital twins, optimization, inverse problems:
  - Time critical: model order reduction and other fast methods
  - Time domain
  - Coupled systems



Polynomial and rational

- Polynomial and rational frequency dependency = linear in the frequency.
- 'Quadratic eigenvalue problem'

$$(K + sC + s^2M)x = f$$

- is 'linearized' to

$$\begin{bmatrix} K & C + sM \\ sI & -I \end{bmatrix} \begin{pmatrix} x \\ sx \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

- Linear in $s = \imath\omega \Longrightarrow$:
  - Time stepping
  - Fast frequency sweeping
  - Eigenvalues

Rational: Plate with poro-elastic damping

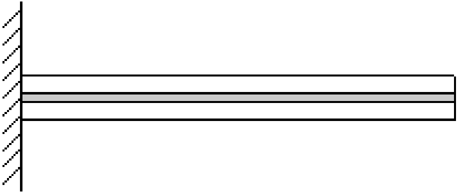
Model:

$$\left(K_e + s^2 M + \left(G_0 + \sum_{j=1}^{p} G_j \frac{s\tau_j}{1 + s\tau_j} \right) K_v \right) x = f$$

with $p = 12$. Problem of size $n = 28,087$ [Lietaert, Deckers, M., 2018]
Linearization:

$$\begin{bmatrix} K_e + G_0 K_v & sM & G_1 K_v & \cdots & G_p K_v \\ sI & -I & & & \\ -s\tau_1 I & & 1 + s\tau_1 & & \\ & \ddots & \ddots & & \\ & & & -s\tau_p & 1 + s\tau_p \end{bmatrix} \begin{pmatrix} x \\ sx \\ \frac{s\tau_1}{1+s\tau_1} x \\ \vdots \\ \frac{s\tau_p}{1+s\tau_p} x \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Nonlinear damping



- Clamped sandwich beam
- Linear system

$$\left(K_e + \frac{G_0 + G_\infty (s\tau)^\alpha}{1 + (s\tau)^\alpha} K_v + s^2 M\right) x = f$$

with $\alpha = 0.675$ and $\tau = 8.230$.
Parameters $G_0, G_\infty, \alpha, \tau$ are obtained from measurements.

Linearizations of nonlinear frequency dependencies

Two approaches for

$$\begin{aligned} A(s)x &= f \\ y &= c^T x \end{aligned}$$

1. Rational approximation of $y(s)$:
   - Sampling methods (Loewner matrices) [Mayo, Antoulas, 2007]
   - Possibly combined with IRKA (TFIRKA) [Beattie, Gugercin, ...]
   - Used for 'matrix free' BEM [Desmet, Jonckheere, 2016]
   - Computational cost is high.
2. Rational approximation of $A(s)$:
   - Approximate $A(s)$ by a (rational) polynomial
   - Form the linear representation
   - All advantages of linear models
   - ... provided the rational approximation is fast to form

Approach of linearization

$$A(s)x = f$$

We assume the following form (holomorphic decomposition):

$$A(s) = \sum_{i=1}^{m} C_i g_i(s)$$

with g_i *holomorphic* in $\imath\mathbb{R}$.
Two steps

1. polynomial/rational approximation

$$A(s) \approx \sum_{i=1}^{m} C_i \psi_i(s)$$

   with ψ_i (rational) polynomial of degree d, with poles outside $\imath\mathbb{R}$.
2. Linearization

Rational approximation

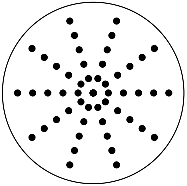
- Padé approximation [Su & Bai, 2011]
- Infinite Arnoldi: Spectral discretization [Trefethen 2000], [Michiels, Niculescu 2007] [Jarlebring, Michiel, M. 2013]
- NLEIGS: potential theory [Güttel, Van Beemen, M. & Michiels, 2014]
- AAA: Adaptive Antoulas Anderson [Nakatsukasa, Sète, Trefethen, 2018] [Lietaert, M., 2018] [Lietaert, M., Perez, Vandereycken, 2020] [Güttel, Negri Porzio and Tisseur, 2020]

AAA approximation

Rational approximation in barycentric form:

$$g(s) \approx r(s) = \sum_{j=1}^{d} \frac{g(z_j)\,\omega_j}{s - z_j} \Big/ \sum_{j=1}^{d} \frac{\omega_j}{s - z_j}$$

Ω



- z_j: support point
- ω_j: weight

Selection of z_j and ω_j: greedy procedure *adaptive Antoulas–Anderson* [Nakatsukasa, Sète, & Trefethen, 2017]

AAA

$$A(s) = A_0 + sB_0 + A_1 g_1(s)$$

$$A(s) \approx R(s) = A_0 + sB_0 + A_1 (a_1^T (E_1 - sF_1)^{-1} b_1$$

and linearization

$$\begin{bmatrix} A_0 + sB_0 & a_1^T \otimes A_1 \\ b_1 \otimes I_n & (E_1 - sF_1) \otimes I_n \end{bmatrix}$$

with

$$\left[\begin{array}{c|c} 0 & a_1^T \\ \hline b_1 & E_1 - sF_1 \end{array}\right] = \left[\begin{array}{c|ccccc} 0 & g_1(z_1) & g_1(z_2) & \cdots & & g_1(z_d) \\ \hline -1 & 1 & 1 & \cdots & & 1 \\ 0 & \omega_2(s - z_1) & \omega_1(z_2 - s) & & & \\ \vdots & & \omega_3(s - z_2) & \ddots & & \\ \vdots & & & \ddots & \omega_{d-2}(z_{d-1} - s) & \\ 0 & & & & \omega_d(s - z_{d-1}) & \omega_{d-1}(z_d - s) \end{array}\right]$$

[Lietaert, M., Pérez, Vandereycken, 2020]

Set valued AAA

$$A(s) = A_0 + sB_0 + \sum_{j=1}^{r} A_j g_j(s)$$

if $r > 1$, then we have to build separate AAA approximations for each g_j and join them together as follows:

$$A(s) = A_0 + sB_0 + \sum_{j=1}^{r} A_j (a_j^T (E_j - sF_j)^{-1} b_j$$

and linearization (for $r = 2$):

$$\begin{bmatrix} A_0 + sB_0 & a_1^T \otimes A_1 & a_2^T \otimes A_2 \\ b_1 \otimes I_n & (E_1 - sF_1) \otimes I_n & 0 \\ b_2 \otimes I_n & 0 & (E_2 - sF_2) \otimes I_n \end{bmatrix}$$

Set valued AAA

Related to [FastAAA by Hochman, 2018]

$$A(s) = A_0 + sB_0 + \sum_{j=1}^{m} A_j g_j(s)$$

Support points and weights are the same for all g_j.

$$A(s) \approx R(s) = A_0 + sB_0 - \sum_{j=1}^{r} (a_j^T \otimes A_j)(b^T (E - sF)^{-1} \otimes I_n)$$

The linearization is

$$\begin{bmatrix} A_0 + sB_0 & \sum_{j=1}^{r} a_j^T \otimes A_j \\ b \otimes I_n & (E - sF) \otimes I_n \end{bmatrix}$$

Set valued AAA

[Elsworth & Güttel, 2018]

Apply AAA to $v^*A(s)u$ for well chosen v and u.

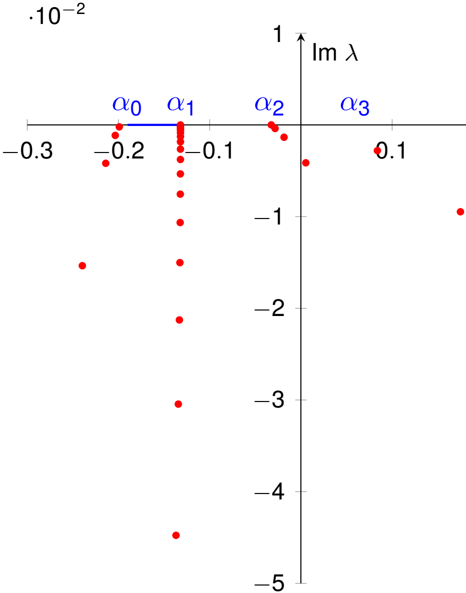
- Use the support points of $v^*A(s)u$ for rational approximation of $A(s)$.
- Good choice when matrices do not have an explicit form $A(s) = \sum_{j=1}^{m} C_j g_j(s)$.
- As matrix vector products $v^*A(s)u$ as test points required.
- We found that g_j are not always well approximated, although the linear combination

$$\sum_{j=1}^{m} (v^* C_j u) g_j(s)$$

is. (Typically lower degree for $v^*A(s)u$.)

Example

1. 2D model of a semiconductor device
2. 81 functions: $g_j = e^{i\sqrt{s-\alpha_j}}$ for $j = 0, \ldots, 80$.
3. interval $[\alpha_0, \alpha_1]$ was discretized with 1000 equidistant interior points.
4. With tolerance 10^{-12} this led to a rational approximation with $d = 45$.

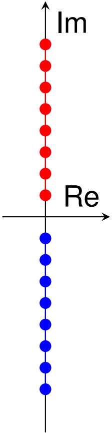


‘Real’ formulation

- Symmetry along the real axis: $g_j(\bar{s}) = \overline{g_j(s)}$.
- Obtain a real valued function for real s.
- Two complex conjugate support points $z_1, z_2 = \overline{z_2}$ (add in pairs):

$$\frac{g(z_1)\omega_1}{s - z_1} + \frac{\overline{g(z_1)}\omega_1}{s - \overline{z_1}} \Big/ \frac{g(z_1)\omega_1}{s - z_1} + \frac{\overline{g(z_1)}\omega_1}{s - \overline{z_1}}$$

- Real weights ω_1 en ω_2.
- Also see [Hochman, 2018], but without linearization.
- Make linearization real valued by linear combination of rows/columns.



Rational Krylov method

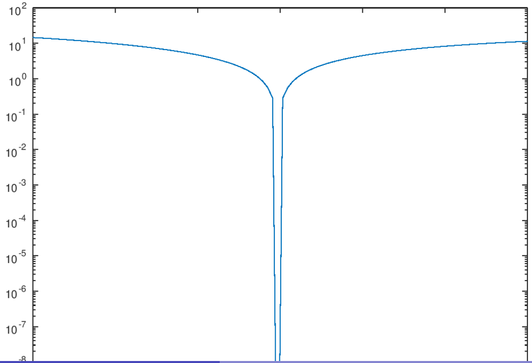
Optimal choice of interpolation points

- IRKA (Iterative Rational Krylov) [Gugercin, Antoulas, Beattie, 2008]
  - Iteratively determine interpolation points that guarantee, on convergence, minimal $\mathcal{H}_2$ error
  - Expensive procedure: each iteration, an order k model has to be constructed
- Greedy optimization [Druskin, Simoncini, 2008] [Druskin, Lieberman, Zaslavsky, 2010]
  - On each iteration, add one interpolation point
  - Choose interpolation point based on an error estimation
  - The easiest is to choose the residual norm of the linear system (cheap and accurate)
  - Does not produce an optimal reduction
- Combination: SPARK [Panzer, Jaensch, Wolf, and Lohmann, 2013].
- Computational improvement: keep the shift during a small number of iterations.

Greedy algorithm

- Build a reduced model of dimension $k \times k$ by projection of the linear model on a subspace
- At iteration k, add state vector $x(\sigma_k)$ with σ_k chosen so that the residual $r(s)$, with $s \in \imath\mathbb{R}$ has largest norm for $s = \sigma_k$.
- Higher order interpolation: build a small Krylov space for the shift:

$$(\sigma_k E - A)^{-1} f, \ldots, ((\sigma_k E - A)^{-1} E)^{m-1} (\sigma_k E - A)^{-1} f$$



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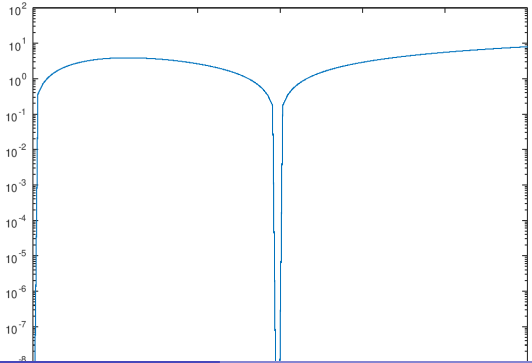

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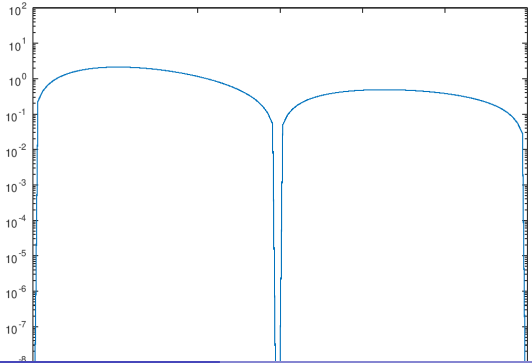

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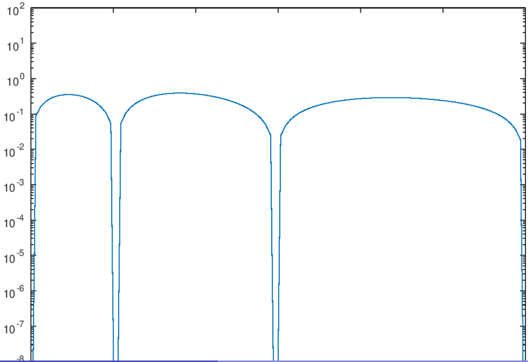

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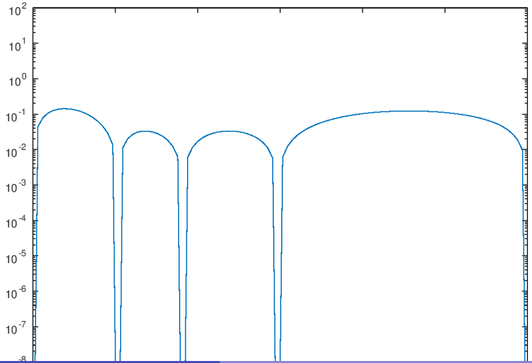

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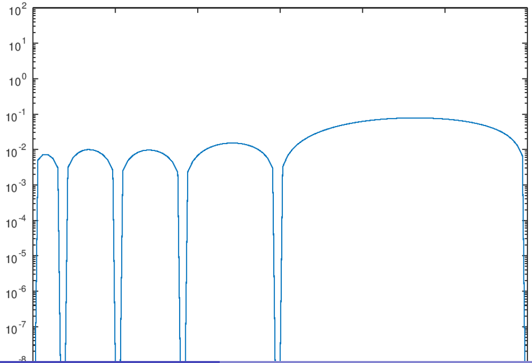

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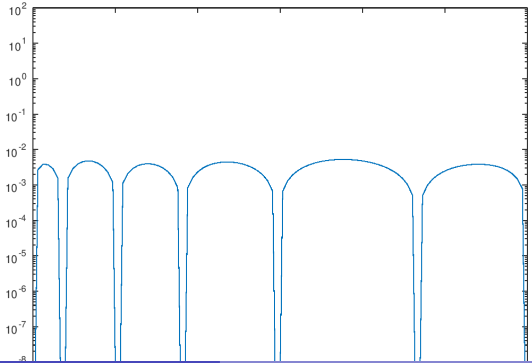

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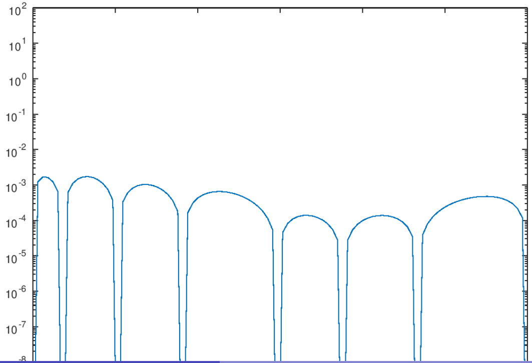

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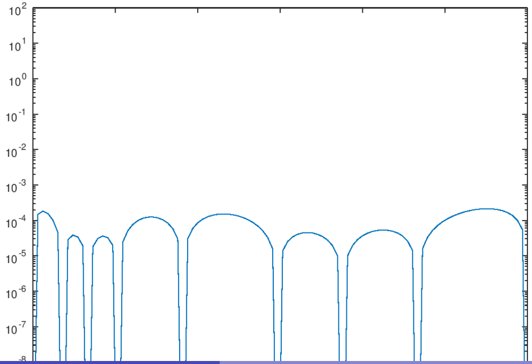

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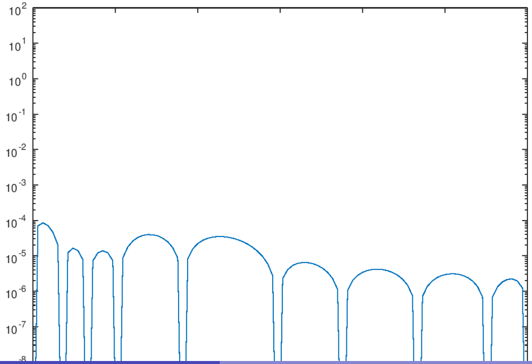


Fast frequency sweeping: plate

- The problem ($n = 28,087$) ($0 - 500$Hz)

$$(K_e + G_v(\omega)K_v - \omega^2 M)x = f$$

$$G_v = G_0 + \sum_{k=1}^{m} G_k \frac{\imath\omega\tau_k}{1 + \imath\omega\tau_k}$$

- Results for Greedy method with multiple shifts:

solves per shift	LU factorizations	subspace dimension	time
1	30	65	251.9
20	4	141	48.1

Time integration

- Linearization in Laplace domain has state vector

$$\begin{pmatrix} x \\ \phi_1 x \\ \vdots \\ \phi_d x \end{pmatrix}$$

  with ϕ_j barycentric rational basis functions.
- In the time domain:

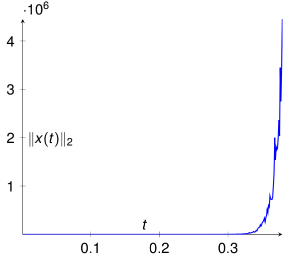
$$\begin{pmatrix} \tilde{x} \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_d \end{pmatrix} \quad \text{with} \quad \tilde{y}_j(t) = \beta_{j,0}\tilde{x}(t) + \sum_{i=1}^{d} \beta_{j,i} \int_0^\infty \tilde{x}(t-s)e^{\alpha_i s} ds.$$

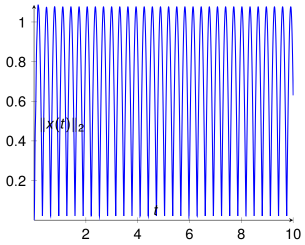
- Initial values: $\tilde{y}_j(0) = 0$ if $x(t) = 0$ for $t \leq 0$ (system in rest position).

Example of the clamped beam

$$\left(K_e + \frac{G_0 + G_\infty (s\tau)^\alpha}{1 + (s\tau)^\alpha} K_v + s^2 M\right) x = 0$$

- right-hand side $f(t) = f_0 \cdot \sin(\omega t)$ with $\omega = 2\pi \cdot 10$.
- Two selections of approximations:
  1. 1000 sample points (log scale) in $[10, 10^3]$Hz: $d = 33$
  2. 1000 sample points (log scale) in $[10, 10^5]$Hz: $d = 47$
- Amplitude corresponds to modulus of transfer function for $s = \imath\omega$.





Conclusions

- Linearizations by AAA leads to fast methods for frequency sweeping and time integration
- Choosing a large frequency range is essential for stability of the ODE/DAE
- Downside: problems with many g_j's to approximate may annihilate the disadvantages.
- Work on going with Simon Dirckx, Daan Huybrechs en Elke Deckers to improve for other situations including BEM.