Semidefinite relaxations in non-convex spaces

Alex Pozas-Kerstjens

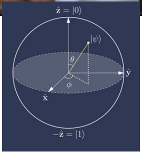
mathQI - Universidad Complutense de Madrid

mathQI

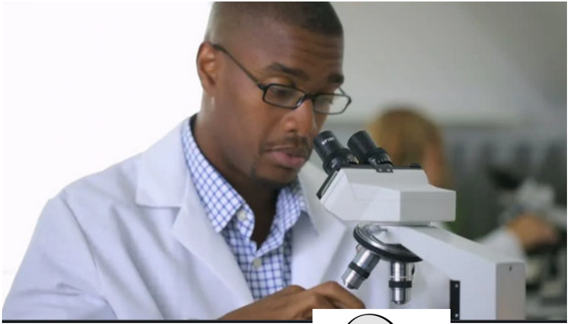
Obra Social "la Caixa"

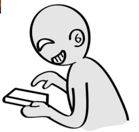
European Research Council
Established by the European Commission

Can I be sure that what I observe is quantum?



Can I be sure that what I observe is quantum?





Quantum models for observations

$p(a, b, \dots | x, y, \dots)$ is quantum $\Rightarrow \exists\, \rho \in \mathcal{H}, \{\Pi^{\mathrm{A}}_{a|x}\}, \{\Pi^{\mathrm{B}}_{b|y}\}, \dots$
such that $p(a, b, \dots | x, y, \dots) = \mathsf{Tr}\left(\rho \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \dots\right)$

Quantum models for observations

$p(a, b, \dots | x, y, \dots)$ is quantum $\Rightarrow \exists\, \rho \in \mathcal{H}, \{\Pi^{\mathrm{A}}_{a|x}\}, \{\Pi^{\mathrm{B}}_{b|y}\}, \dots$
such that $p(a, b, \dots | x, y, \dots) = \mathsf{Tr}\left(\rho \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \dots\right)$

$\rho \succeq 0,\ \mathsf{Tr}(\rho) = 1,$

Quantum models for observations

$p(a, b, \dots | x, y, \dots)$ is quantum $\Rightarrow \exists\, \rho \in \mathcal{H}, \{\Pi^{\mathrm{A}}_{a|x}\}, \{\Pi^{\mathrm{B}}_{b|y}\}, \dots$
such that $p(a, b, \dots | x, y, \dots) = \mathrm{Tr}\left(\rho \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \dots\right)$

$\rho \succeq 0,\ \mathrm{Tr}(\rho) = 1,$ $\qquad \Pi^{\mathrm{P}}_{o|i} \succeq 0,\ \sum_o \Pi^{\mathrm{P}}_{o|i} = \mathbb{1},$

Quantum models for observations

$p(a,b,\dots|x,y,\dots)$ is quantum $\Rightarrow \exists\, \rho \in \mathcal{H}, \{\Pi^{\mathrm{A}}_{a|x}\}, \{\Pi^{\mathrm{B}}_{b|y}\}, \dots$
such that $p(a,b,\dots|x,y,\dots) = \mathsf{Tr}\left(\rho \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \dots\right)$

$\rho \succeq 0$, $\mathsf{Tr}(\rho) = 1$, $\qquad \Pi^{\mathrm{P}}_{o|i} \succeq 0$, $\sum_o \Pi^{\mathrm{P}}_{o|i} = \mathbb{1}$,
$\Pi^{\mathrm{P}}_{o|i} \Pi^{\mathrm{P}}_{o'|i} = \delta_{o,o'} \Pi^{\mathrm{P}}_{o|i}$,

Quantum models for observations

$p(a, b, \dots | x, y, \dots)$ is quantum $\Rightarrow \exists\, \rho \in \mathcal{H}, \{\Pi^{\mathrm{A}}_{a|x}\}, \{\Pi^{\mathrm{B}}_{b|y}\}, \dots$
such that $p(a, b, \dots | x, y, \dots) = \mathsf{Tr}\left(\rho \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \dots\right)$

$\rho \succeq 0$, $\mathsf{Tr}(\rho) = 1$, $\quad \Pi^{\mathrm{P}}_{o|i} \succeq 0$, $\sum_o \Pi^{\mathrm{P}}_{o|i} = \mathbb{1}$,
$\Pi^{\mathrm{P}}_{o|i} \Pi^{\mathrm{P}}_{o'|i} = \delta_{o,o'} \Pi^{\mathrm{P}}_{o|i}$, $\quad \Pi^{\mathrm{P}}_{o|i} \Pi^{\mathrm{P}'}_{o'|i'} = \Pi^{\mathrm{P}'}_{o'|i'} \Pi^{\mathrm{P}}_{o|i}$ if $\mathrm{P} \neq \mathrm{P}'$.

Quantum models for observations

$p(a,b,\dots|x,y,\dots)$ is quantum $\Rightarrow \exists\, \rho \in \mathcal{H}, \{\Pi^{\mathrm{A}}_{a|x}\}, \{\Pi^{\mathrm{B}}_{b|y}\}, \dots$
such that $p(a,b,\dots|x,y,\dots) = \mathsf{Tr}\left(\rho \Pi^{\mathrm{A}}_{a|x}\Pi^{\mathrm{B}}_{b|y}\dots\right)$

$\rho \succeq 0$, $\mathsf{Tr}(\rho) = 1$, $\quad \Pi^{\mathrm{P}}_{o|i} \succeq 0$, $\sum_o \Pi^{\mathrm{P}}_{o|i} = \mathbb{1}$,
$\Pi^{\mathrm{P}}_{o|i}\Pi^{\mathrm{P}}_{o'|i} = \delta_{o,o'}\Pi^{\mathrm{P}}_{o|i}$, $\quad \Pi^{\mathrm{P}}_{o|i}\Pi^{\mathrm{P}'}_{o'|i'} = \Pi^{\mathrm{P}'}_{o'|i'}\Pi^{\mathrm{P}}_{o|i}$ if $\mathrm{P} \neq \mathrm{P}'$.

$\Rightarrow \forall \mathcal{S} = \{\Pi^{\mathrm{P}}_{o|i}, \Pi^{\mathrm{P}_1}_{o_1|i_1}\Pi^{\mathrm{P}_2}_{o_2|i_2}, \dots\}$
$\Gamma|\Gamma_{i,j} = \mathsf{Tr}\left(\rho S_i^\dagger S_j\right)$, $S_i, S_j \in \mathcal{S}$
$\Gamma \succeq 0$

The NPA hierarchy

$\mathcal{S}_n = \{$all products of operators of length $\leq n\}$

p is quantum $\Leftrightarrow \Gamma^{(n)} \succeq 0 \, \forall n$

p is not quantum $\Rightarrow \exists n^*$ such that $\Gamma^{(n \geq n^*)} \prec 0$

M. Navascués, S. Pironio, A. Acín, New J. Phys. 10, 073013 (2008)

The NPA hierarchy

$\mathcal{S}_n = \{$all products of operators of length $\leq n\}$
p is quantum $\Leftrightarrow \Gamma^{(n)} \succeq 0 \, \forall n$
p is not quantum $\Rightarrow \exists n^*$ such that $\Gamma^{(n \geq n^*)} \prec 0$

In practice:

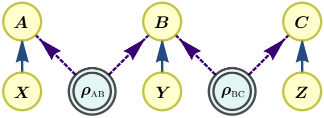
$$p(a, b, c, \dots | x, y, z, \dots) \stackrel{?}{=} \mathrm{Tr}(\rho \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \dots)$$

1. Choose symbols that define $\mathcal{S} = \{\Pi^{\mathrm{P}}_{o|i}, \Pi^{\mathrm{P}_1}_{o_1|i_1} \Pi^{\mathrm{P}_2}_{o_2|i_2}, \dots\}$
2. Create $\Gamma | \Gamma_{i,j} = \mathrm{Tr}\left(\rho S_i^\dagger S_j\right)$, $S_i, S_j \in \mathcal{S}$
3. Solve the SDP

$$[\text{find vars / optimize } f] \text{ s.t. } \Gamma \succeq 0$$

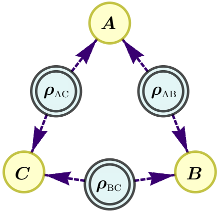
M. Navascués, S. Pironio, A. Acín, New J. Phys. 10, 073013 (2008)

Restricting the set of possible ρ



$$\rho = \rho_{\text{AB}} \otimes \rho_{\text{BC}}$$

$$\sum_b p(a,b,c|x,y,z) = p(a|x)p(c|z)$$



$$\rho = \rho_{\text{AB}} \otimes \rho_{\text{BC}} \otimes \rho_{\text{CA}}$$

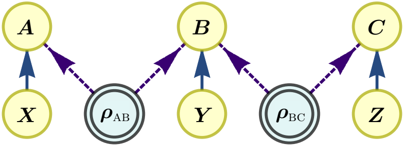
$\Gamma^{(n)} \succeq 0 \, \forall \, n$ if p is quantum,
but the feasible regions are not convex

Two SDP-compatible methods

Scalar extension (Phys. Rev. Lett. 123, 140503)
– Independence constraints
– “Experimentally” powerful

Quantum inflation (Phys. Rev. X 11, 021043)
– More general non-convex spaces
– Precise notion of hierarchy

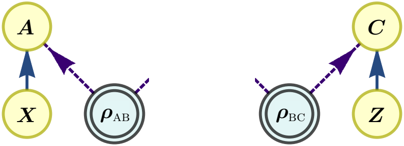
Scalar extension for independence constraints



$$p(a,b,c|x,y,z) = \mathsf{Tr}\left(\rho_{AB_1} \otimes \rho_{B_2C} \Pi^{A}_{a|x} \Pi^{B}_{b|y} \Pi^{C}_{c|z}\right)$$

Phys. Rev. Lett. 123, 140503 (2019)

Scalar extension for independence constraints



$$p(a,b,c|x,y,z) = \mathsf{Tr}\left(\rho_{\mathrm{AB}_1} \otimes \rho_{\mathrm{B}_2\mathrm{C}} \Pi^{\mathrm{A}}_{a|x} \Pi^{\mathrm{B}}_{b|y} \Pi^{\mathrm{C}}_{c|z}\right)$$

$$\sum_b p(a,b,c|x,y,z) = p(a|x)p(c|z)$$

$$\left\langle \Pi^{\mathrm{A}}_{a_1} \ldots \Pi^{\mathrm{A}}_{a_k} \Pi^{\mathrm{C}}_{c_1} \ldots \Pi^{\mathrm{C}}_{c_l} \right\rangle = \left\langle \Pi^{\mathrm{A}}_{a_1} \ldots \Pi^{\mathrm{A}}_{a_k} \right\rangle \left\langle \Pi^{\mathrm{C}}_{c_1} \ldots \Pi^{\mathrm{C}}_{c_l} \right\rangle$$

Phys. Rev. Lett. 123, 140503 (2019)

Scalar extension for independence constraints

$$\left\langle \Pi^{\mathrm{A}}_{a_1} \dots \Pi^{\mathrm{A}}_{a_k} \Pi^{\mathrm{C}}_{c_1} \dots \Pi^{\mathrm{C}}_{c_l} \right\rangle = \left\langle \Pi^{\mathrm{A}}_{a_1} \dots \Pi^{\mathrm{A}}_{a_k} \right\rangle \left\langle \Pi^{\mathrm{C}}_{c_1} \dots \Pi^{\mathrm{C}}_{c_l} \right\rangle$$

Complement sets of generating monomials with operator strings *multiplied by scalars*.

Phys. Rev. Lett. 123, 140503 (2019)

Scalar extension for independence constraints

$$\left\langle \Pi^{\mathrm{A}}_{a_1} \ldots \Pi^{\mathrm{A}}_{a_k} \Pi^{\mathrm{C}}_{c_1} \ldots \Pi^{\mathrm{C}}_{c_l} \right\rangle = \left\langle \Pi^{\mathrm{A}}_{a_1} \ldots \Pi^{\mathrm{A}}_{a_k} \right\rangle \left\langle \Pi^{\mathrm{C}}_{c_1} \ldots \Pi^{\mathrm{C}}_{c_l} \right\rangle$$

Complement sets of generating monomials with operator strings *multiplied by scalars*.

Example: $\mathcal{S} = \{\mathbb{1}, A_0 A_1, C_0 C_1, \langle A_0 A_1 \rangle \mathbb{1}\}$

Phys. Rev. Lett. 123, 140503 (2019)

Scalar extension for independence constraints

$$\left\langle \Pi^{\mathrm{A}}_{a_1} \dots \Pi^{\mathrm{A}}_{a_k} \Pi^{\mathrm{C}}_{c_1} \dots \Pi^{\mathrm{C}}_{c_l} \right\rangle = \left\langle \Pi^{\mathrm{A}}_{a_1} \dots \Pi^{\mathrm{A}}_{a_k} \right\rangle \left\langle \Pi^{\mathrm{C}}_{c_1} \dots \Pi^{\mathrm{C}}_{c_l} \right\rangle$$

Complement sets of generating monomials with operator strings *multiplied by scalars*.

Example: $\mathcal{S} = \{\mathbb{1}, A_0A_1, C_0C_1, \langle A_0A_1\rangle\, \mathbb{1}\}$

$$\Gamma = \begin{array}{c} \\ \mathbb{1} \\ (A_0A_1)^\dagger \\ (C_0C_1)^\dagger \\ \langle A_0A_1\rangle^* \mathbb{1} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbb{1} & A_0A_1 & C_0C_1 & \langle A_0A_1\rangle \mathbb{1} \end{array} \\ \begin{pmatrix} 1 & {\color{red}v_1} & v_2 & {\color{red}v_1} \\ & 1 & {\color{magenta}v_3} & {\color{blue}v_4} \\ & & 1 & {\color{green}v_5} \\ & & & {\color{blue}v_4} \end{pmatrix} \end{array}$$

Phys. Rev. Lett. 123, 140503 (2019)

Scalar extension for independence constraints

$$\left\langle \Pi^{\mathrm{A}}_{a_1} \ldots \Pi^{\mathrm{A}}_{a_k} \Pi^{\mathrm{C}}_{c_1} \ldots \Pi^{\mathrm{C}}_{c_l} \right\rangle = \left\langle \Pi^{\mathrm{A}}_{a_1} \ldots \Pi^{\mathrm{A}}_{a_k} \right\rangle \left\langle \Pi^{\mathrm{C}}_{c_1} \ldots \Pi^{\mathrm{C}}_{c_l} \right\rangle$$

Complement sets of generating monomials with operator strings *multiplied by scalars*.

Example: $\mathcal{S} = \{\mathbb{1}, A_0A_1, C_0C_1, \langle A_0A_1\rangle \mathbb{1}\}$

$$\Gamma = \begin{array}{c} \\ \mathbb{1} \\ (A_0A_1)^\dagger \\ (C_0C_1)^\dagger \\ \langle A_0A_1\rangle^* \mathbb{1} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbb{1} & A_0A_1 & C_0C_1 & \langle A_0A_1\rangle \mathbb{1} \end{array} \\ \begin{pmatrix} 1 & v_1 & v_2 & v_1 \\ & 1 & v_3 & v_4 \\ & & 1 & v_3 \\ & & & v_4 \end{pmatrix} \end{array}$$

$$\langle A_0A_1C_0C_1\rangle = \langle A_0A_1\rangle \langle C_0C_1\rangle$$

Phys. Rev. Lett. 123, 140503 (2019)

Scalar extension for independence constraints

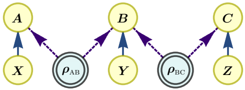
$$\Gamma = \begin{array}{c} \mathbb{1} \\ (A_0A_1)^\dagger \\ (C_0C_1)^\dagger \\ \langle A_0A_1\rangle^*\mathbb{1} \end{array} \overset{\begin{array}{cccc} \mathbb{1} & A_0A_1 & C_0C_1 & \langle A_0A_1\rangle\mathbb{1} \end{array}}{\begin{pmatrix} 1 & v_1 & v_2 & v_1 \\ & 1 & v_3 & v_4 \\ & & 1 & v_3 \\ & & & v_4 \end{pmatrix}} \succeq 0$$

$$\Downarrow$$

$$\begin{aligned} -1 &\leq v_1 \leq 1 \\ -1 &\leq v_2 \leq 1 \\ v_1^2 &\leq v_4 \leq 1 \\ v_1v_2 - \sqrt{(v_4 - v_1^2)(1 - v_2^2)} &\leq v_3 \leq v_1v_2 + \sqrt{(v_4 - v_1^2)(1 - v_2^2)} \end{aligned}$$

Phys. Rev. Lett. 123, 140503 (2019)

Approximations are tight in certain situations



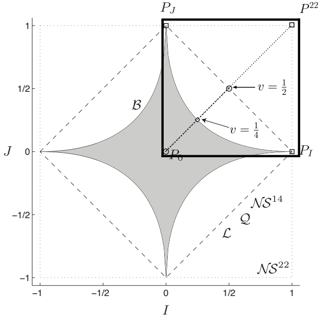
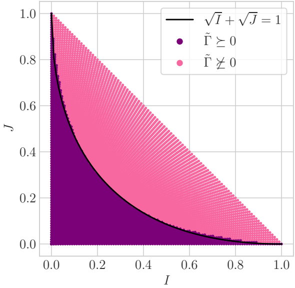


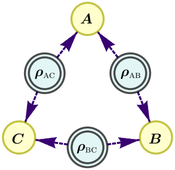
Image from
Phys. Rev. A 85, 032119 (2012)



$\mathcal{S}_3 \cup \{\langle A_0 A_1 \rangle \mathbb{1}\}$

A.P.-K. PhD Thesis, http://hdl.handle.net/10803/667696

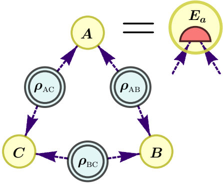
Quantum inflation for subtler nonconvexities



Phys. Rev. X 11, 021043 (2021)

Quantum inflation for subtler nonconvexities

$$\rho = \rho_{\mathrm{AB}} \otimes \rho_{\mathrm{BC}} \otimes \rho_{\mathrm{AC}}$$

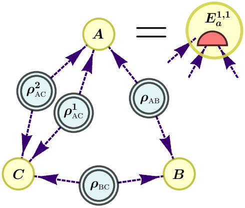


$$p(a,b,c) = \left\langle E_a F_b G_c \right\rangle_\rho$$

Phys. Rev. X 11, 021043 (2021)

Quantum inflation for subtler nonconvexities

$$\rho = \rho_{\mathrm{AB}} \otimes \rho_{\mathrm{BC}} \otimes \rho^1_{\mathrm{AC}} \otimes \rho^2_{\mathrm{AC}}$$

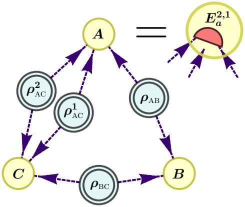


$$p(a,b,c) = \left\langle E_a^{1,1} F_b G_c^{1,1} \right\rangle_\rho$$

Phys. Rev. X 11, 021043 (2021)

Quantum inflation for subtler nonconvexities

$$\rho = \rho_{\mathrm{AB}} \otimes \rho_{\mathrm{BC}} \otimes \rho^1_{\mathrm{AC}} \otimes \rho^2_{\mathrm{AC}}$$

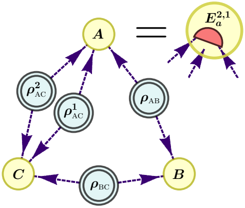


$$\begin{aligned} p(a,b,c) &= \langle E_a^{1,1} F_b G_c^{1,1} \rangle_\rho \\ &= \langle E_a^{2,1} F_b G_c^{1,2} \rangle_\rho \end{aligned}$$

Phys. Rev. X 11, 021043 (2021)

Quantum inflation for subtler nonconvexities

$$\rho = \rho_{\mathrm{AB}} \otimes \rho_{\mathrm{BC}} \otimes \rho^1_{\mathrm{AC}} \otimes \rho^2_{\mathrm{AC}}$$

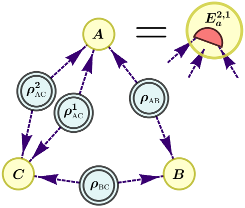


$$p(a,b,c) = \left\langle E_a^{1,1} F_b G_c^{1,1} \right\rangle_\rho = \left\langle E_a^{2,1} F_b G_c^{1,2} \right\rangle_\rho$$

$$\left\langle Q(\{E_a^{1,1}\},\{E_a^{2,1}\},\{F_b\},\{G_c^{1,1}\},\{G_c^{1,2}\}) \right\rangle_\rho = \left\langle Q(\{E_a^{2,1}\},\{E_a^{1,1}\},\{F_b\},\{G_c^{1,2}\},\{G_c^{1,1}\}) \right\rangle_\rho$$

Phys. Rev. X 11, 021043 (2021)

Quantum inflation for subtler nonconvexities



$\exists\ \rho$

such that

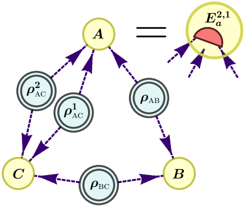
$$p(a,b,c) = \left\langle E_a^{1,1} F_b G_c^{1,1} \right\rangle_\rho = \left\langle E_a^{2,1} F_b G_c^{1,2} \right\rangle_\rho$$

$$\left\langle Q(\{E_a^{1,1}\},\{E_a^{2,1}\},\{F_b\},\{G_c^{1,1}\},\{G_c^{1,2}\}) \right\rangle_\rho = \left\langle Q(\{E_a^{2,1}\},\{E_a^{1,1}\},\{F_b\},\{G_c^{1,2}\},\{G_c^{1,1}\}) \right\rangle_\rho ?$$

Can be solved with the NPA hierarchy

Phys. Rev. X 11, 021043 (2021)

Quantum inflation for subtler nonconvexities



$\exists\ \rho$

such that

$$p(a,b,c) = \left\langle E_a^{1,1} F_b G_c^{1,1} \right\rangle_\rho = \left\langle E_a^{2,1} F_b G_c^{1,2} \right\rangle_\rho$$

$$\left\langle Q(\{E_a^{1,1}\},\{E_a^{2,1}\},\{F_b\},\{G_c^{1,1}\},\{G_c^{1,2}\}) \right\rangle_\rho = \left\langle Q(\{E_a^{2,1}\},\{E_a^{1,1}\},\{F_b\},\{G_c^{1,2}\},\{G_c^{1,1}\}) \right\rangle_\rho ?$$

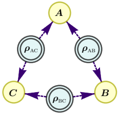
Can be solved with the NPA hierarchy

Inflation hierarchy ($m = \#$ copies) + NPA hierarchy for each m

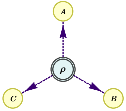
---

Phys. Rev. X 11, 021043 (2021)

Application 1: Constrained optimization



vs.



Mermin

$$\langle A_1B_0C_0\rangle + \langle A_0B_1C_0\rangle + \langle A_0B_0C_1\rangle - \langle A_1B_1C_1\rangle \leq \begin{cases} \mathbf{3.085}^* & \mathcal{Q}^{\triangle} \\ 4 & \mathcal{Q} \end{cases}$$

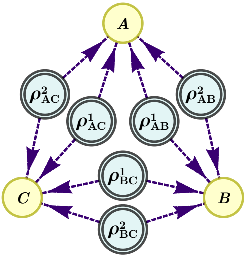
Svetlichny

$$\begin{aligned}&\langle A_1B_0C_0\rangle + \langle A_0B_1C_0\rangle + \langle A_0B_0C_1\rangle - \langle A_1B_1C_1\rangle \\ &\quad -\langle A_0B_1C_1\rangle - \langle A_1B_0C_1\rangle - \langle A_1B_1C_0\rangle + \langle A_0B_0C_0\rangle \leq \begin{cases} \mathbf{4.405}^* & \mathcal{Q}^{\triangle} \\ 4\sqrt{2} & \mathcal{Q} \end{cases}\end{aligned}$$

Phys. Rev. X 11, 021043 (2021)

Application 2: Optimization of polynomial objectives

Example: distance to a given distribution p_{t}



$$f(p) = \sum_{a,b,c} |p(a,b,c) - p_{\mathsf{t}}(a,b,c)|^2$$

$$= \sum_{a,b,c} \Big[p(a,b,c)p(a,b,c) - 2p_{\mathsf{t}}(a,b,c)p(a,b,c) \Big] + \text{const}$$

$$= \sum_{a,b,c} \Big[\big\langle E_a^{1,1} E_a^{2,2} F_b^{1,1} F_b^{2,2} G_c^{1,1} G_c^{2,2} \big\rangle_\rho - 2p_{\mathsf{t}}(a,b,c) \big\langle E_a^{1,1} F_b^{1,1} G_c^{1,1} \big\rangle_\rho \Big] + \text{const}$$

Phys. Rev. X 11, 021043 (2021)

Summary

Goal: Semidefinite relaxations of non-convex feasible spaces

Summary

Goal: Semidefinite relaxations of non-convex feasible spaces

	Scalar extension	Quantum inflation
Main idea	NPA + new symbols for nonlinear operators	Copies of the scenario + NPA
Good for	Factorizations	Latent tensor products
Strengths	More efficient Probably more useful outside QI	(Much) more general Optimization of polynomial objectives
Weaknesses	Optimization of polynomial objectives	Computationally demanding

Summary

Goal: Semidefinite relaxations of non-convex feasible spaces

	Scalar extension	Quantum inflation
Main idea	NPA + new symbols for nonlinear operators	Copies of the scenario + NPA
Good for	Factorizations	Latent tensor products
Strengths	More efficient Probably more useful outside QI	(Much) more general Optimization of polynomial objectives
Weaknesses	Optimization of polynomial objectives	Computationally demanding
Open questions	- Convergence (arXiv:2006.12510) - Smart choice of $\mathcal{S}$	- Convergence (arXiv:1707.06476, arXiv:1911.11056) - Smart choice of $\mathcal{S}$ - Hierarchy balance

Thank you for your attention

Questions? Comments?

1904.08943 / PRL 123, 140503 (2019) Scalar Extension
1909.10519 / PRX 11, 021043 (2021) Quantum Inflation

physics@alexpozas.com

apozas/quantum-networks-scalar-extension

mathQI

European Commission

Horizon 2020
European Union funding
for Research & Innovation

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 648913)