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PORTOROŽ SLOVENIA

Transitions between configurations

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Introductory remarks

In this talk I would like to briefly review some constructions which can be used to obtain new configurations from old configurations.

Some of such constructions go back to as early as the 2nd half of the 19th century, but my aim is much more modest, and I restrict myself to developments in the last several decades.

Even in this case it is far from being a complete survey. All these constructions will be presented through illustrating examples, rather than, or besides, by a formal description.

Part of the results mentioned here were obtained in joint work with Tomo Pisanski, Leah Berman, Nino Bašić, Marko Boben, and Jurij Kovič.

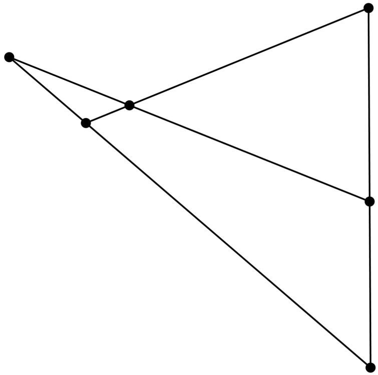
Binary operations

- Introduced formally with this name by GG & Pisanski, 2014, but used intuitively by several other authors, too.
- Informally speaking, it is a superposition of two configurations, supplemented by new incidences.

Definition

By the incidence sum of configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ we mean the configuration $\mathcal{C}$ which is the disjoint union of $\mathcal{C}_1$ and $\mathcal{C}_2$, together with a specified set $I \subseteq P_1 \times L_2 \cup P_2 \times L_1$ of incident point-line pairs, where P_i denotes the point set and L_i denotes the line set of $\mathcal{C}_i$, for $i = 1, 2$. We denote it by $\mathcal{C}_1 \oplus \mathcal{C}_2$.

Example 1. [Boben, GG & Pisanski, 2015]



Complete quadrilateral $(6_2, 4_3)$

Binary operations – (1) Incidence sum

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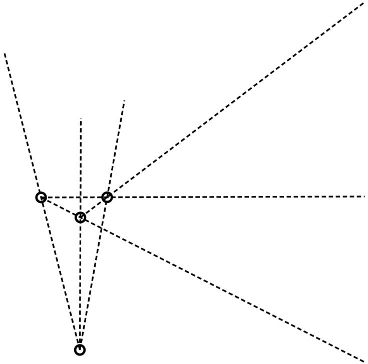

Complete quadrangle $(4_3, 6_2)$

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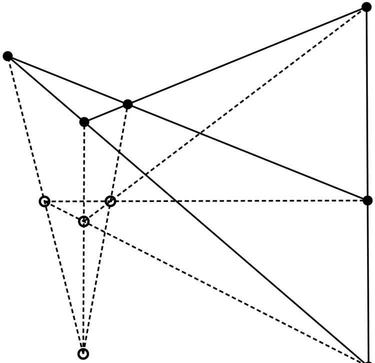

Desargues configuration: $(10_3) \cong (6_2, 4_3) \oplus (4_3, 6_2)$

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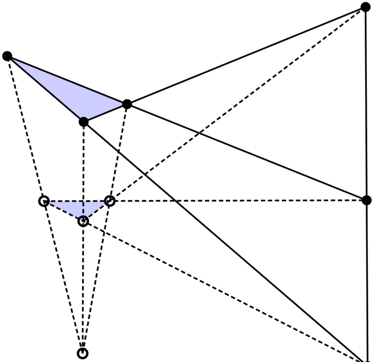

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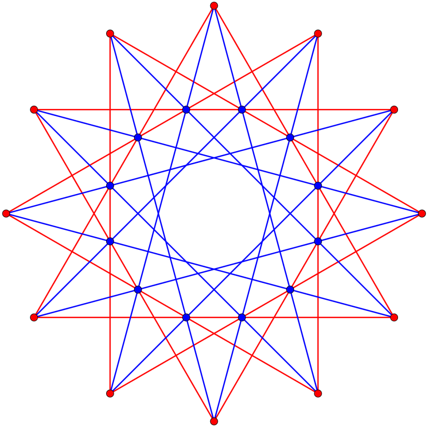
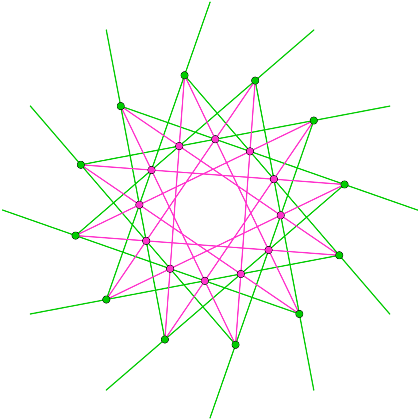


$\mathcal{C}$: the 2-astral configuration $12\#(4,1;4,5)$; type: (24_4)

Example 2. [Berman & Ng, 2010]



$\mathcal{C}'$: a homothetic copy of $\mathcal{C}$

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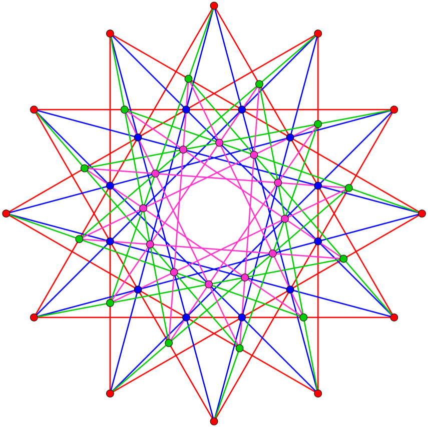

$\mathcal{C} \oplus \mathcal{C}'$. Type (48_5)

Example 3. [GG, 2018]

$$\left(\binom{3}{2}_2, \binom{3}{1}_2\right)$$

$$\left(\binom{4}{2}_2, \binom{4}{1}_3\right) \quad \left(\binom{4}{3}_3, \binom{4}{2}_2\right)$$

$$\left(\binom{5}{2}_2, \binom{5}{1}_4\right) \quad \left(\binom{5}{3}_3, \binom{5}{2}_3\right) \quad \left(\binom{5}{4}_4, \binom{5}{3}_2\right)$$

$$\left(\binom{6}{2}_2, \binom{6}{1}_5\right) \quad \left(\binom{6}{3}_3, \binom{6}{2}_4\right) \quad \left(\binom{6}{4}_4, \binom{6}{3}_3\right) \quad \left(\binom{6}{5}_5, \binom{6}{4}_2\right)$$

$$\left(\binom{7}{2}_2, \binom{7}{1}_6\right) \quad \left(\binom{7}{3}_3, \binom{7}{2}_5\right) \quad \left(\binom{7}{4}_4, \binom{7}{3}_4\right) \quad \left(\binom{7}{5}_5, \binom{7}{4}_3\right) \quad \left(\binom{7}{6}_6, \binom{7}{5}_2\right)$$

$$\vdots$$

Pascal's triangle of Desargues–Cayley–Danzer configurations

Binary operations – (1) Incidence sum

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"Cayley–Dickson line" [Saniga, Holweck & Pracna, 2015]

Binary operations – (2) Cartesian product

The notion of a Cartesian product of configurations has been introduced independently

- on the combinatorial level by Pisanski & Servatius, 2013, and
- for geometric point-line configurations by GG, 2014.

Definition

Let $\mathcal{C}$ be configuration of type (v_r, b_k) and $\mathcal{C}'$ a configuration of type $(v'_{r'}, b'_k)$. Observe that these two configurations have the same number k of points in each block. The Cartesian product of $\mathcal{C}$ and $\mathcal{C}'$ is a configuration of type

$$\left((vv')_{(r+r')}, (vb' + v'b)_k\right),$$

whose point set is the Cartesian product of the point sets of $\mathcal{C}$ and $\mathcal{C}'$ and where there is a block incident to two points (x, x') and (y, y') if and only if either $x = y$ and there is a block incident to x' and y' in $\mathcal{C}'$, or $x' = y'$ and there is a block incident to x and y in $\mathcal{C}$.

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- If the factors $\mathcal{C}$, $\mathcal{C}'$ are embedded in space of dimension d and d', then the product occurs in space of dimension $d + d'$.

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Example. An infinite sequence based on complete *n*-laterals:

- complete 5-lateral: $(10_2, 5_4)^2 = (100_4) \subset \mathbb{P}^4$
- complete 7-lateral: $(21_2, 7_6)^3 = (9261_6) \subset \mathbb{P}^6$
- complete 9-lateral: $(36_2, 9_8)^4 = (1679616_8) \subset \mathbb{P}^8$
- complete 11-lateral: $(55_2, 11_{10})^5 = (503284375_{10}) \subset \mathbb{P}^{10}$

$\vdots$

(A complete *n*-lateral is a configuration consisting of *n* lines in general position and of their $\binom{n}{2}$ intersection points.)

Binary operations – (2) Cartesian product

For plane configurations, it can also be realized in the same plane, by means of the Minkowski addition [GG, Bašić, Kovič & Pisanski]:

- the point set of the product is the Minkowski sum of the point sets of the component configurations:

$$P(\mathcal{C}_1 \otimes \mathcal{C}_2) = \{x + y \mid x \in P(\mathcal{C}_1), y \in P(\mathcal{C}_2)\}.$$

- the new blocks are obtained as translates of the blocks of the one component by the position vectors of the points of the other component, and vice versa:

$$B(\mathcal{C}_1 \otimes \mathcal{C}_2) = \{\tau_y B(\mathcal{C}_1) \mid y \in P(\mathcal{C}_2)\} \cup \{\tau_x B(\mathcal{C}_2) \mid x \in P(\mathcal{C}_1)\}.$$

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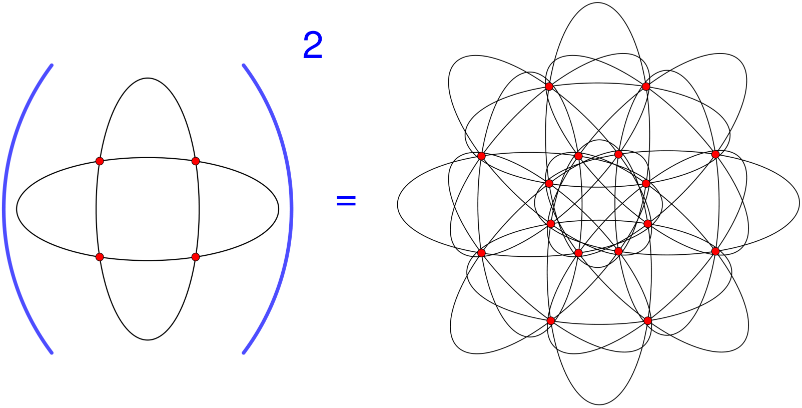

$(4_2, 2_4)$ (16_4)

Incidence switch

"Switch" from old to new incidences [Grünbaum, 2009; Pisanski & Servatius, 2013]

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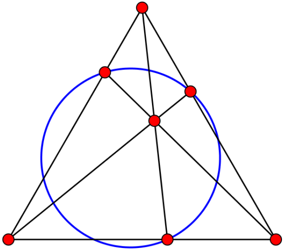
Example 1. Applied to the Fano configuration:



(7_3)

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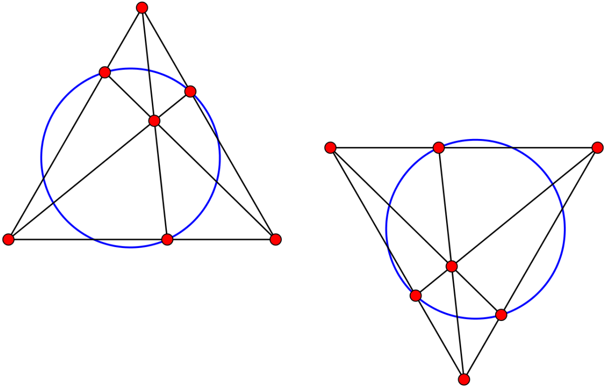

$2 \times (7_3)$

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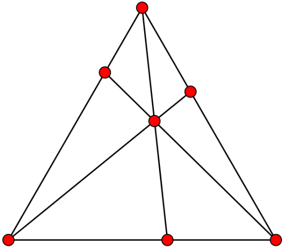

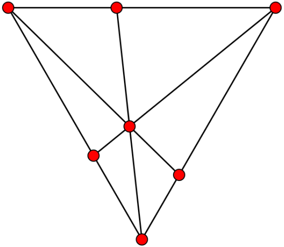

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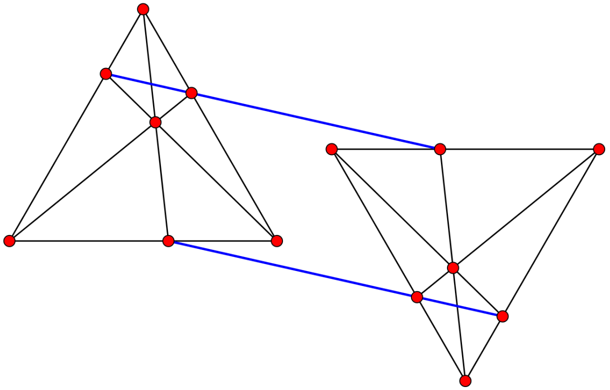

(14_3)

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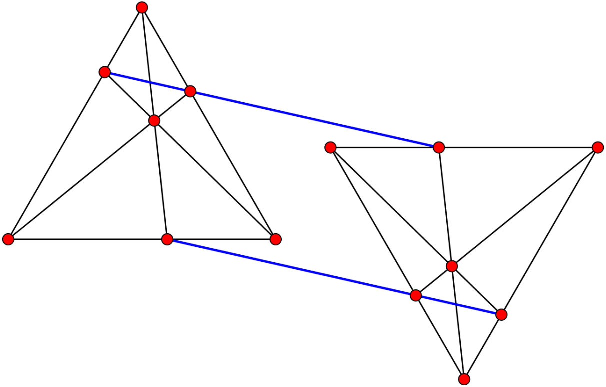

(14_3) (Realizable !)

Remark.

- In this example, we started from a non-realizable configuration, and we obtained a realizable new configuration.

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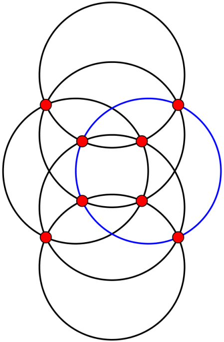
- In this example, we started from a non-realizable configuration, and we obtained a realizable new configuration.
- The converse case also occurs: if a configuration is connected to an incidence theorem, then the incidence switch results in a non-realizable configuration.

Example 2a. Applied to the Miquel configuration
[GG, Bašić, Kovič & Pisanski, 2021]



$(8_3, 6_4)$

Example 2a. Applied to the Miquel configuration
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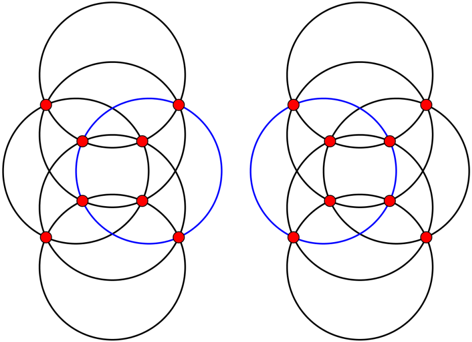

$2 \times (8_3, 6_4)$

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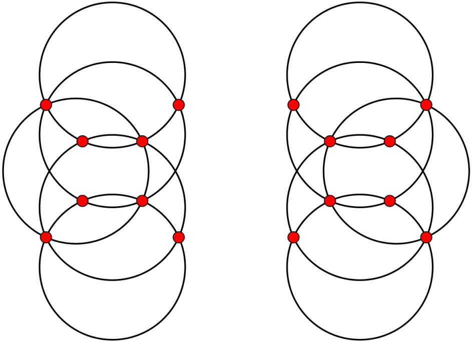

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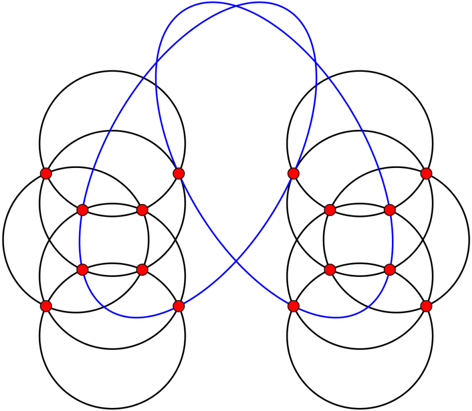
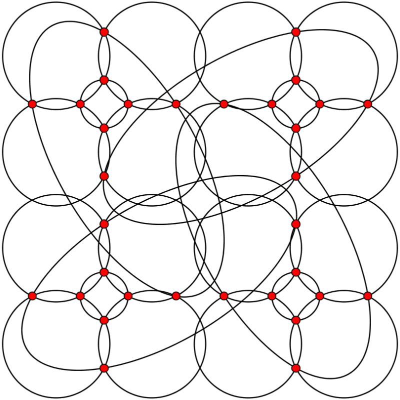


$(16_3, 12_4)$

Example 2b. Applied as a quaternary operation
[GG, Bašić, Kovič & Pisanski, 2021]



$(32_3, 24_4)$

Grünbaum incidence calculus

The "Grünbaum incidence calculus" is the common name of a collection of constructions elaborated by Branko Grünbaum;

- presented in detail in [Grünbaum, 2009] (and in some earlier works of him);
- discussed also in [Pisanski & Servatius, 2013] (they also coined the name "Grünbaum calculus");
- some of them are modified and generalized in [Berman, GG & Pisanski, 2021].

Grünbaum incidence calculus

- Parallel switch: $(p_q, n_k) \longrightarrow \big((kp)_q, (kn)_k\big)$
  - uses parallel translations of the starting configuration;
  - the incidence numbers q, k are preserved.

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- Affine switch: starting from an (m_k) configuration with independent pencils of $p \geq 0$ and $q \geq 1$ parallel lines, for each integer r with $1 \leq r \leq p+q$, it produces a configuration of type $\big((k-1)m+r\big)$.
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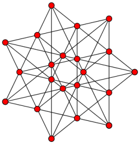
  - uses affine transforms of the starting configuration.
- Affine replication: $(n_k) \longrightarrow \big(((k+2)n)_{k+1}\big)$;
  - uses affine transforms of the starting configuration.
- Deleted union construction: $(m_k) + (n_k) \longrightarrow \big((m+n-1)_k\big)$;
  - uses projective geometric relationships.

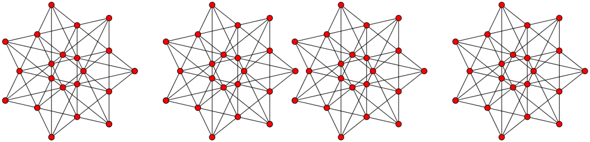
Combining the parallel switch and the deleted union construction



(21_4)

Grünbaum incidence calculus

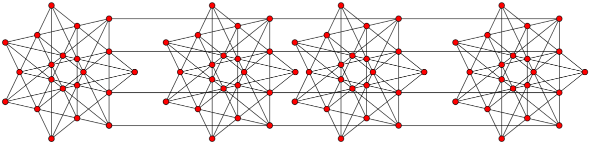
Combining the parallel switch and the deleted union construction



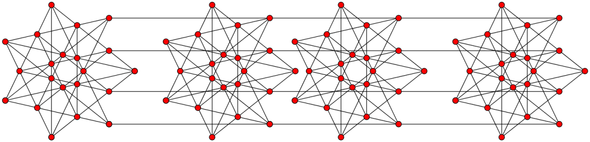
$4 \times (21_4)$

Grünbaum incidence calculus

Combining the parallel switch and the deleted union construction



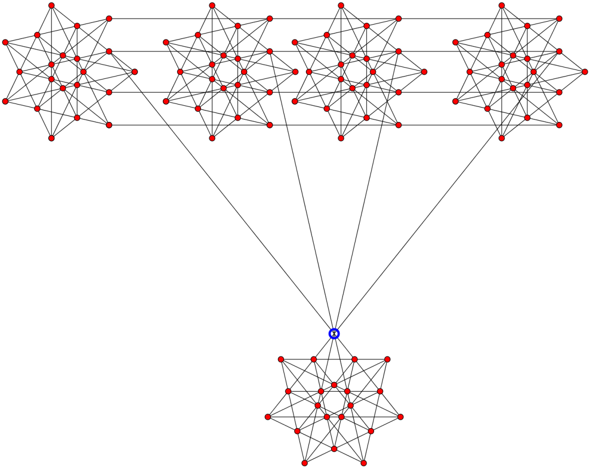
Combining the parallel switch and the deleted union construction



(84_4)

Grünbaum incidence calculus

Combining the parallel switch and the deleted union construction



$(84_4) + (21_4)$

Grünbaum incidence calculus

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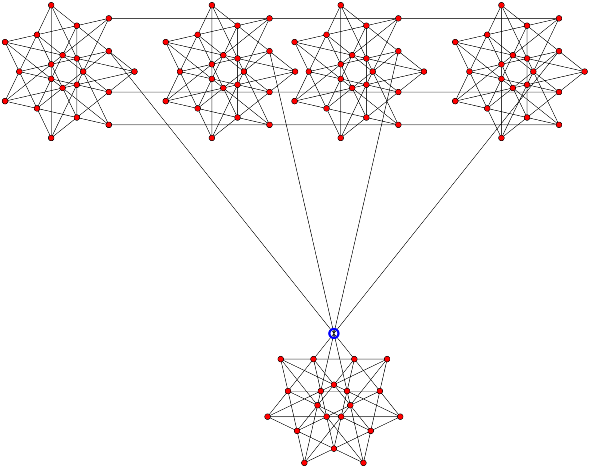

Grünbaum incidence calculus

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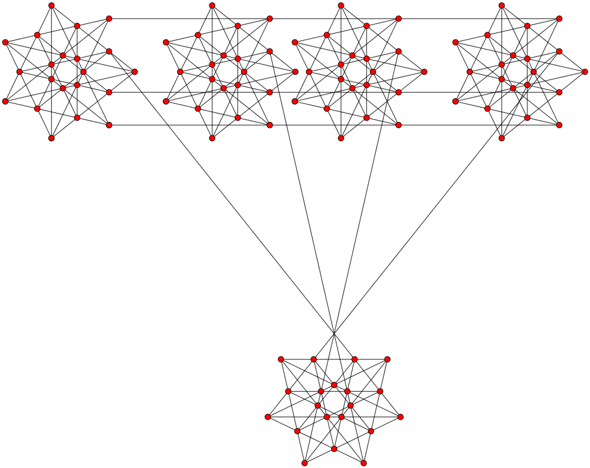
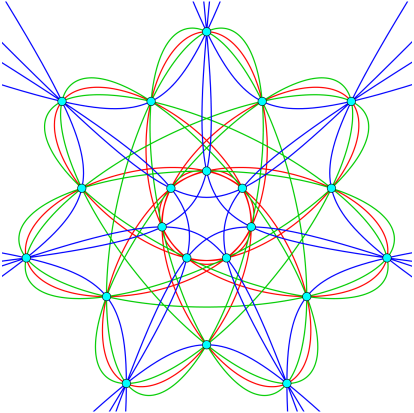
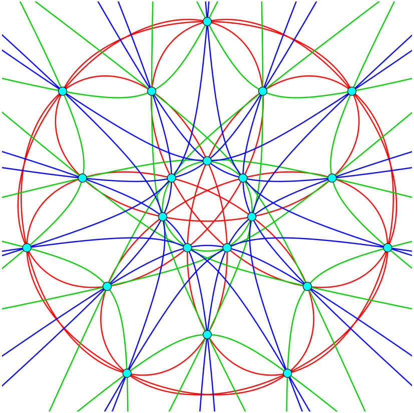


(104_4)

Example 1. Producing a pair of (21_7) configurations [GG, 2019]



(21_7)-EEH



(21_7)-EHH

Transitions: replacing the lines of a configuration by conics

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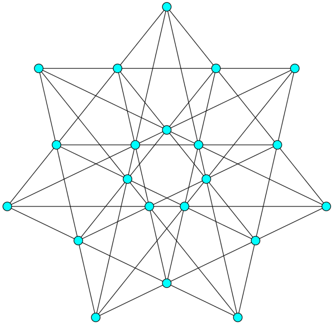
Start from the Grünbaum–Rigby configuration of type (21_4)



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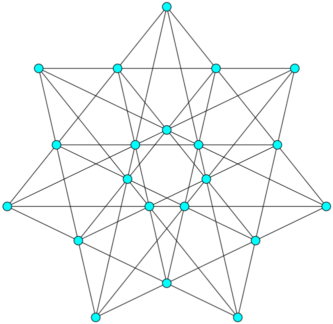

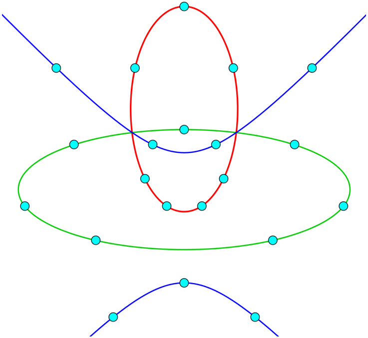

An observation by Luis Montejano: conics can be circumscribed around suitable 7-tuples of points of this configuration.

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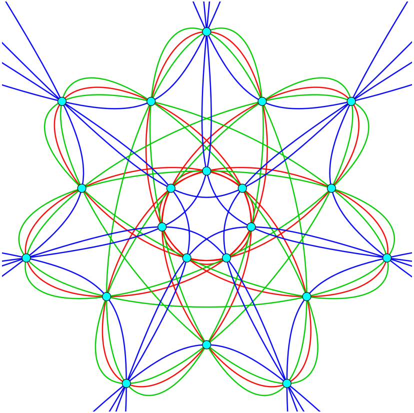
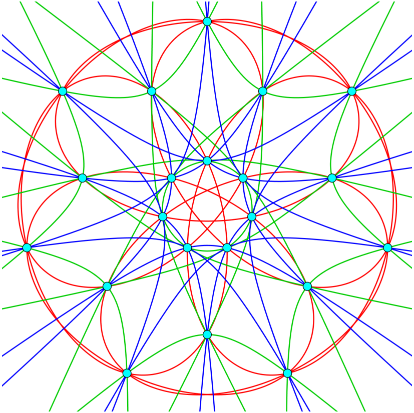


(21_7)-EEH



(21_7)-EHH

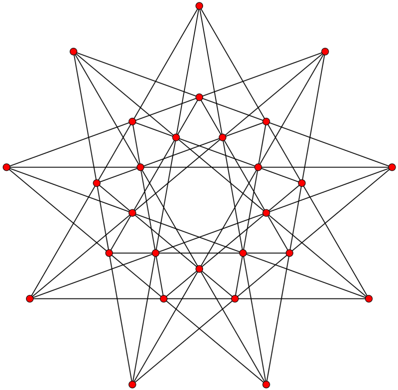
Example 1. Producing a pair of (21_7) configurations [GG, 2019]

Remark. Recall that the lines and points of the (21_4) Grünbaum–Rigby configuration correspond to the axes and centres of the 21 harmonic homologies within the automorphism group of the Klein quartic

$$x^3y + y^3z + z^3x = 0.$$

Question. Can the conics in the two (21_7) configurations be related in some direct way to the Klein quartic?

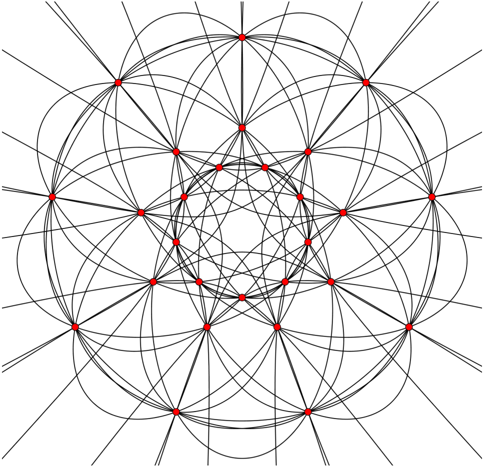
Example 2. A family of type (27_8)



Start from a point-line configuration of type (27_4)
Grünbaum notation: $9\#(4,3;2,3;1,3)$ [Grünbaum, 2009]

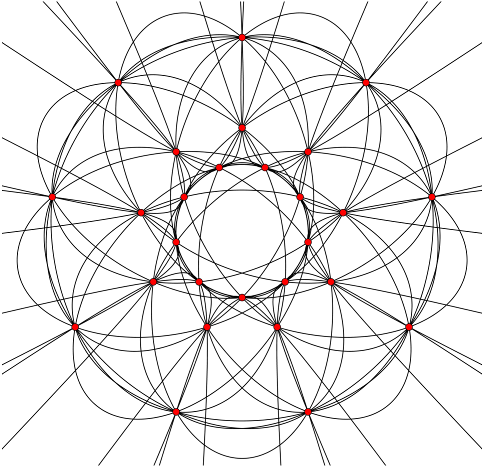
Transitions: replacing the lines of a configuration by conics

Example 2. A family of type (27_8)



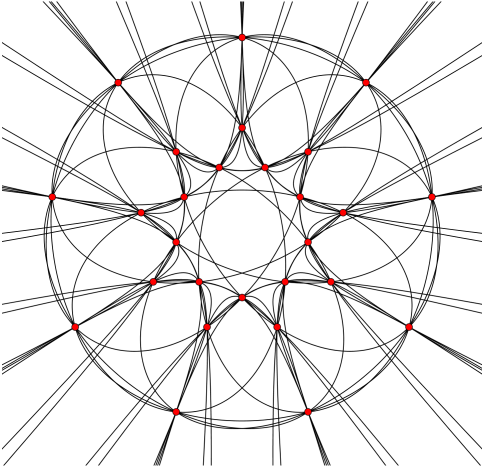
(27_8)-$E_1 E_3 H_3$

Example 2. A family of type (27_8)



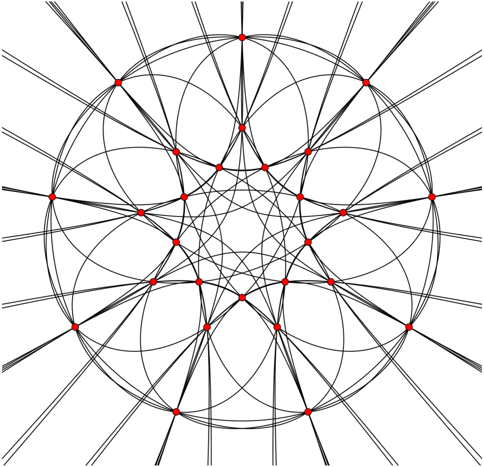
(27_8)-$E_1 H_2 E_3$

Example 2. A family of type (27_8)



(27_8)-$E_1H_2H_4$

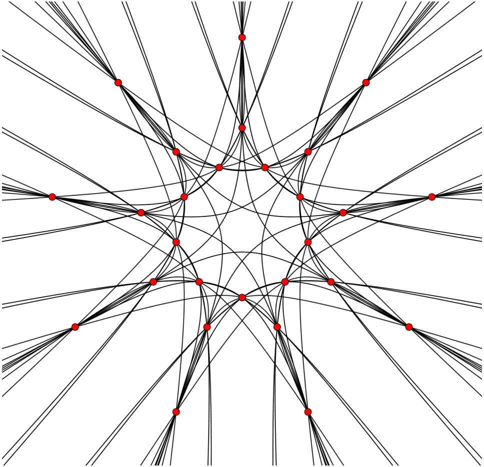
Example 2. A family of type (27_8)



(27_8)-$E_1 H_3 H_4$

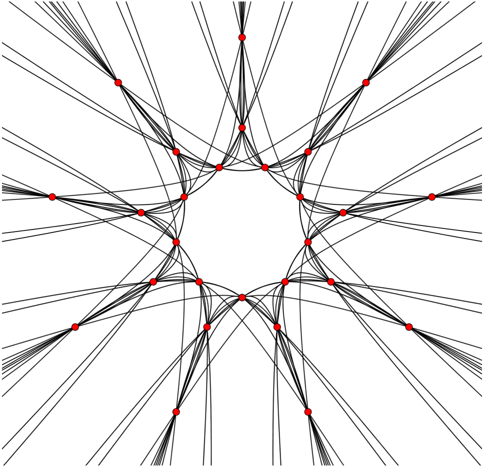
Transitions: replacing the lines of a configuration by conics

Example 2. A family of type (27_8)



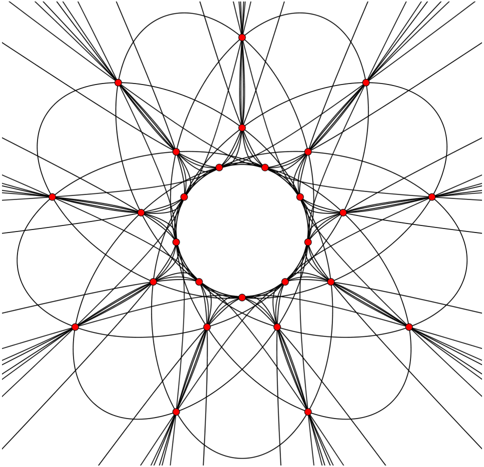
(27_8)-$H_1 H_3 H_4$

Example 2. A family of type (27_8)



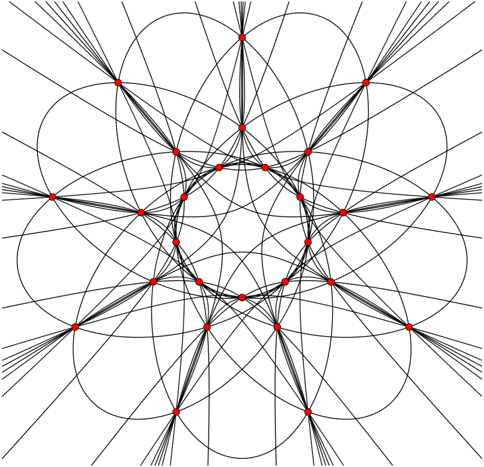
(27_8)-$H_1 H_2 H_4$

Example 2. A family of type (27_8)



(27_8)-$H_1 H_2 E_3$

Example 2. A family of type (27_8)



(27_8)-$H_1 E_3 H_3$

Transitions: isomorphic transmutation

"Transmutation": changing the shape of the blocks in a configuration while preserving the incidences.

(Some examples in [GG and Pisanski, 2014].)

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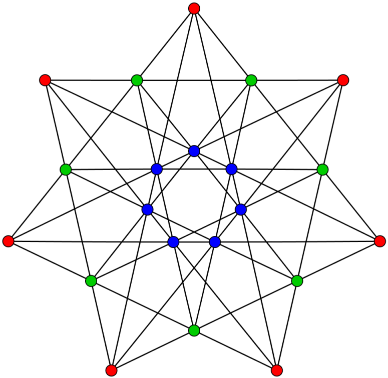
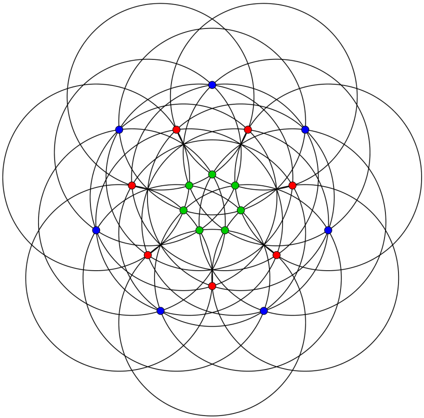
(Some examples in [GG and Pisanski, 2014].)

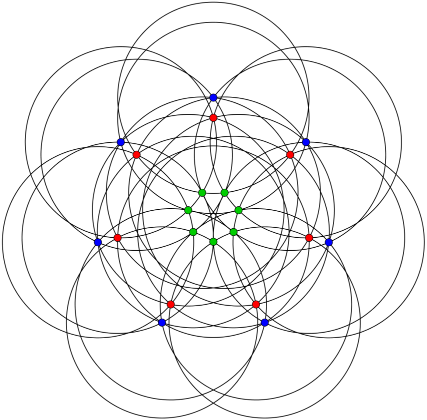
Example. The (21_4) Grünbaum–Rigby configuration.



Example. The (21_4) Grünbaum–Rigby configuration:
isometric point-circle representation.

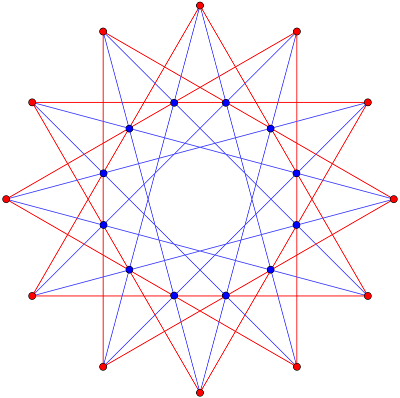


Version A

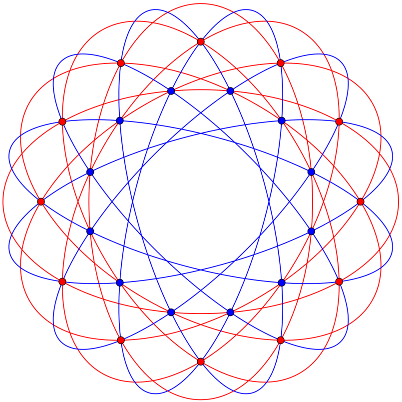


Version B

Example: Transition from type (24_4) to type $(24_4, 12_8)$



(24_4)



$(24_4, 12_8)$

Transitions: non-isomorphic transmutation

Example: Transition from type (24_4) to type $(24_4, 12_8)$

- It can be combined with Cartesian squaring:

$$\begin{array}{ccc} (24_4) & \longrightarrow & (24_4, 12_8) \\ \downarrow & & \downarrow \\ (576_8, 1152_4) & \longrightarrow & (576_8) \end{array}$$

Transitions: radical axes and centres of point-circle configurations

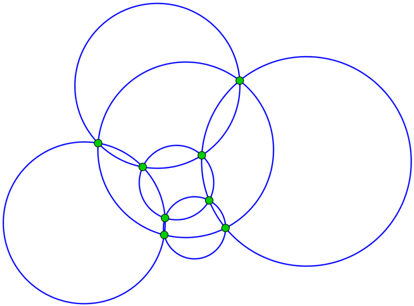
Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.

Construction:

- start from the $(8_3, 6_4)$ Miquel configuration of points and circles;
- take the radical axis for each pair of circles;
- take the radical centre for each triple of circles.

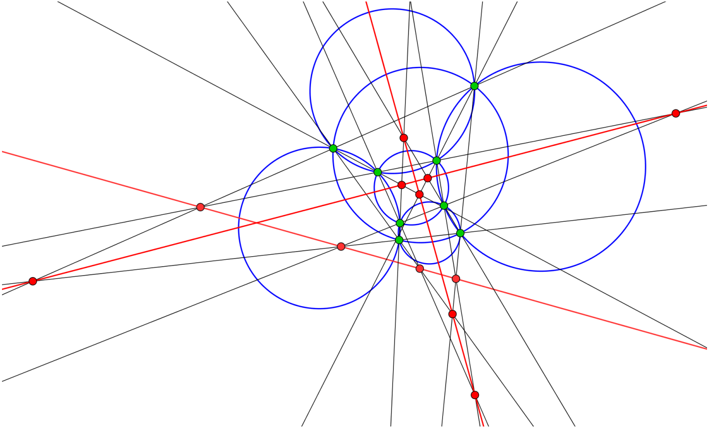
We obtain a point-line configuration of type $(20_3, 15_4)$ which is isomorphic to the Steiner–Plücker configuration.

Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.



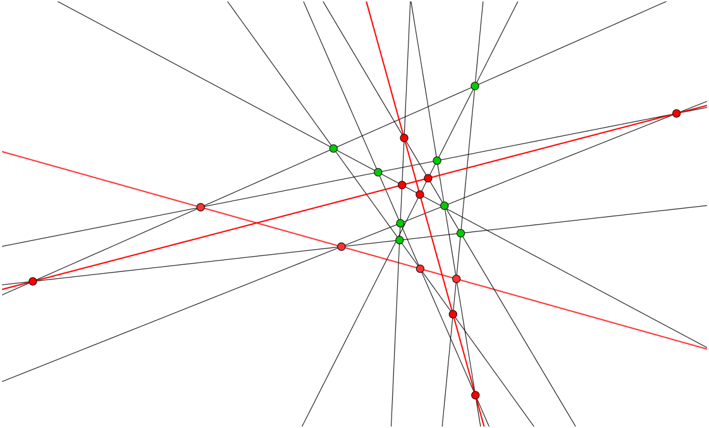
The $(8_3, 6_4)$ Miquel configuration

Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.



Take the radical axes and radical centres

Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.



The $(20_3, 15_4)$ Steiner–Plücker configuration

A generalization:

- start from a point-circle configuration of type $\big((kn)_3, (3n)_k\big)$ $(k, n \geq 3)$ such that no four circles have the same radical centre;
- the construction yields a point-line configuration of type

$$\left(\binom{3n}{3}_3, \binom{3n}{2}_{3n-2}\right).$$

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[10] Pisanski, T. and Servatius, B., *Configurations from a Graphical Viewpoint*, Birkhäuser, Boston, 2013.

End

Thank you for
your attention.