Complexity of Graph Manifolds

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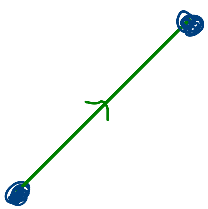
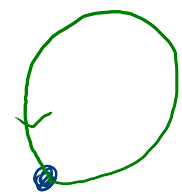




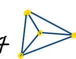
A. Cattabriga, M. Mulazzani, The complexity of orientable graph manifolds, *Adv. Geom.*, accepted for publication (2021).

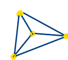
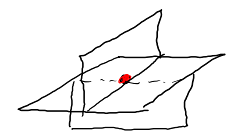
1. Complexity

Assumption: M closed 3-manifold.

Definition: The complexity $c(M)$ of M is the minimum number of true vertices among all almost simple spines of M. (Matveev '90)

Spine: polyhedron P embedded in M s.t. $M \setminus P \cong$ open 3-ball.

Almost simple: the link of each point can be embedded in K_4 

True vertex: a point whose link is K_4  ⇝ locally Butterfly 

Properties: → $c(M_1 \# M_2) = c(M_1) + c(M_2)$

→ the number of prime manifolds having complexity k is finite, $\forall k \in \mathbb{Z}$ $k \geq 0$.

⇝ catalogue http://matlas.math.csu.ru/?page=search

$k=0$: S^3, $\mathbb{RP}^3$, $L(3,1)$, $S^2 \times S^1$

$k=2$: first Seifert (prism manifold) not lens space (total 4)

$k=7$: first graph not Seifert (total 175)

$k=9$: first hyperbolic manifold (total 1154)

$k=12$: 23434 manifolds!

2. Graph Manifolds

Definition: A <u>GRAPH MANIFOLD</u> IS AN ORIENTABLE 3-MANIFOLD OBTAINED BY GLUING SEIFERT MANIFOLDS ALONG TORIC BOUNDARY COMPONENTS. (WALDHAUSEN '67)

(V,E) finite non trivial connected digraph

$V \ni v \longrightarrow S_v$ Seifert $d(v) = \#$ cmp's ∂S_v

$E \ni e \longrightarrow \begin{pmatrix} \alpha_e & \beta_e \\ \gamma_e & \delta_e \end{pmatrix} \in GL_2^-(\mathbb{Z})$

$S(\Sigma^{\pm}_{g,d}, \underbrace{(p_1,q_1), (p_2,q_2), \dots, (p_r,q_r)}_{\text{exceptional fibers},\ 0<q_i<p_i}, \underbrace{b}_{})$

$S \to \Sigma^{\pm}_{g,d}$ = base space

$\bigcirc \to \cdot$

exceptional fibers

$0 < q_i < p_i$

ordinary Euler class

$b \leftrightarrow (1,b)$

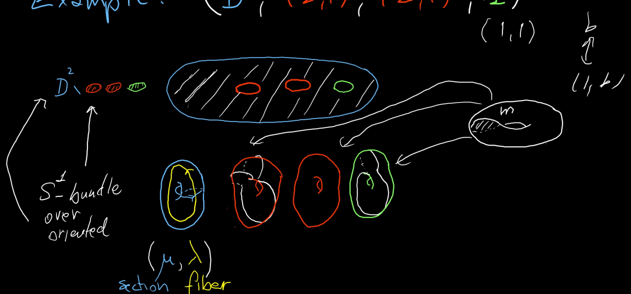
$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$S(D^2, (2,1), (2,1), -1) \xrightarrow{A} S(D^2, (2,1), (2,1), 0)$

$\mu \longrightarrow 2\mu' + \lambda'$

$\lambda \longrightarrow \mu + \lambda$

Example: $(D^2, (2,1), (2,1), 1)$



3. Graph manifolds: complexity upper bounds

Lens space
(Matveev '90)

$$c(L(p,q)) \leq \max\{S(p/q)-3, 0\}$$

$S(p/q) = a_1 + a_2 + \cdots + a_s$

$$p/q = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_s}}}$$

(Jaco - Rubinstein - Tillmann '09 - '11)

$$c(L(2n,1)) = 2n-3, \quad n \geq 2 \qquad c(L(4n, 2n-1)) = n \quad n \geq 1$$

Seifert manifolds
(Martelli - Petronio '04
C. - Matveev - Mulazzani
Nasybullov '20)

Closed, orientable, prime, connected, not lens spaces.

$$c\left(S\left(\Sigma, (p_1,q_1), (p_2,q_2), \cdots, (p_k,q_k), b\right)\right) \leq 6(1-\chi) + \sum_{k=1}^{r}\left(S(p_k/q_k)+1\right) + \max\{0, b-1+\chi\}$$

Graph manifolds
(C. - Mulazzani '21)

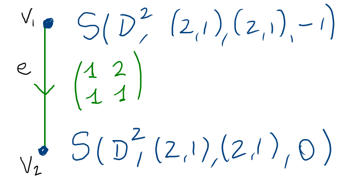
Closed, orientable, connected, not Seifert or torus bundles over S^1.

$$\begin{aligned} c(M_{(V,E)}) \leq\ & 5(|E|-|V|+1) + \Phi(G) + \sum_{e\in E''}(S(\beta_e/\delta_e)-1) + \\ & + \sum_{v\in V}\left(3(d_v+2h_v-2) + \sum_{k=1}^{r_v}(S(p_k/q_k)+1)\right) + \\ & + \min_{T\in\mathcal{O}_G}\left\{\min_{\psi\in\Psi_T,\psi'\in\Psi'_T}\{\sum_{v\in V} f_{m_v,M_v}(b_v)\}\right\}, \end{aligned}$$

Sharp for all (14502) manifolds in the catalogue.

4. Toward the general formula

STEP 1) Tree with no edge labelled $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



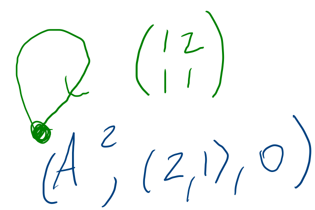
$$c(M) \leq \sum_{e\in E}(S(\beta_e/\delta_e)-1) + \sum_{v\in V}\left(3(d_v+2h_v-2)+\sum_{k=1}^{r_v}(S(p_k/q_k)+1)+f_{m_v,M_v}(b_v)\right)$$

$m_v = 1 - (\#\text{exc. fibers} + \#\text{handles} + \#\text{edges in})$

$M_v = (\#\text{handles} + \#\text{edges out}) - 1$

STEP 2) Graph with no edge labelled $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$c(M) \leq 5(|E|-|V|+1) + \sum_{e\in E}(S(\beta_e/\delta_e)-1) + \sum_{v\in V}\left(3(d_v+2h_v-2)+\sum_{k=1}^{r_v}(S(p_k/q_k)+1)+f_{m_v,M_v}(b_v)\right)$$

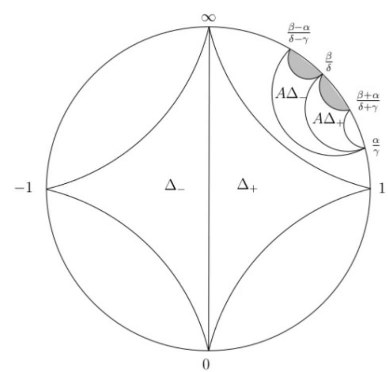


STEP 3) The general case

$$\begin{aligned} c(M_{(V,E)}) \leq\ & 5(|E|-|V|+1)+\Phi(G)+\sum_{e\in E''}(S(\beta_e/\delta_e)-1)+ \\ & +\sum_{v\in V}\left(3(d_v+2h_v-2)+\sum_{k=1}^{r_v}(S(p_k/q_k)+1)\right)+ \\ & +\min_{T\in\mathcal{O}_G}\left\{\min_{\psi\in\Psi_T,\psi'\in\Psi'_T}\{\sum_{v\in V} f_{m_v,M_v}(b_v)\}\right\}, \end{aligned}$$

where:

$$m_v = -r_v-h_v-d_v^- - d^-_{v,\psi,T} - d^-_{v,\psi',T}+1,\ M_v = h_v+d_v^+ + d^+_{v,\psi,T}+d^+_{v,\psi',T}-1$$



Thanks

for your

attention!