Highly symmetric configurations

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Contents

1. Basic notions

A **combinatorial configuration** $C = (V, B, I)$
is a triple of finite sets, whose elements are called

- "points",
- "blocks", and
- "incidence relations" between points and blocks.

A **geometric configuration** $\mathcal{C}$ has points and lines in some n-dimensional Euclidean space or in some other linear space.

A **geometric representation** $\mathcal{C}(C)$ of C is any other "visualisation" of C (e.g. with points and circles or ellipses).

A "**monovalent**" (p_q, n_k) configuration consists of

- p points incident with q blocks
- q blocks incident with k points

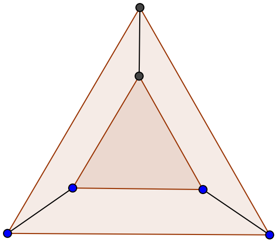
Note that $pq = nk$ (= the number of incidences).

A **balanced** (n_k) configuration consists of

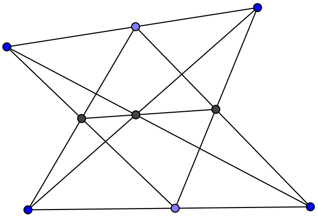
- n k-valent points
- n k valent blocks.

Example

A monovalent unbalanced configuration and
a balanced "Pappus" configuration.



$(6_3, 9_2)$

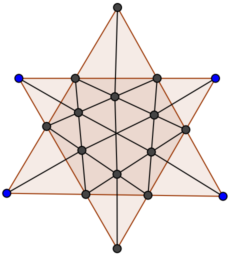


(9_3)

For "**polivalent**" configurations we use a natural generalization of this notation:

Example

$C = (6_4 12_3, 15_4)$ has 6 points of valence 4, 12 points of valence 3 and 15 lines of valence 4.



A "**valence structure**" notation of the same configuration is: $(4_6^{15} 3_{12})$.

Automorphisms and symmetries

We distinguish between

- **automorphisms** of a combinatorial configuration C and
- **symmetries** of its various geometric realisations $\mathcal{C}$:

- automorphisms preserve the algebraic structure of C, while
- symmetries are those isometries (e.g. rotations, reflections, translation, etc.) of the ambiental space S which preserve $\mathcal{C}$.

In general, the symmetry group is smaller:

$\mathrm{Sym}(\mathcal{C}) \trianglelefteq Aut(C)$.

Given C, how to represent as many automorphisms of C as possible as symmetries of $\mathcal{C}$?

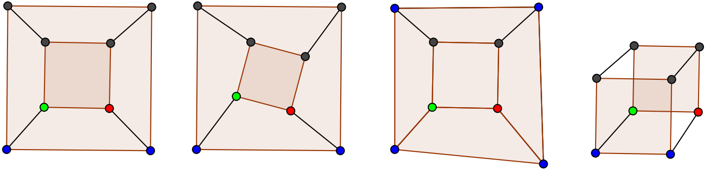
The symmetry group $Sym(\mathcal{C})$ of $\mathcal{C}$ depends on the

- **type and dimension** n **of the space** S in which C is embedded as a geometrical object $\mathcal{C}$ (e.g. Euclidean, projective, hyperbolic, etc.); the higher is n the more automorphisms of C we may actually "see" as symmetries of $\mathcal{C}$
- **type of geometric objects by which we represent** C (e.g. with lines, circles, ellipses, spheres, spyrals, etc.)
- **embedding itself** (an infinitesimally small change of $\mathcal{C}$ may "break" some or all the symmetries)

Planar and spatial configurations

Symmetries of a (finite) planar geometrical configuration $\mathcal{C} \subset E^2$ may be only **rotations** and **reflections**, thus $Sym(\mathcal{C})$ must be a subset of some **cyclical group** $\mathbb{Z}_m$ or some **dihedral group** D_m.

The same combinatorial configuration C may have a bigger symmetry group, if it is realised as a spatial configuration in some higher-dimensional space.



2. How to construct highly symmetric configurations?

There are many methods, techniques and tricks for how to do it, for example:

Method 1. Use symmetric objects

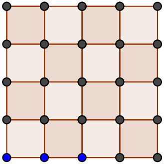
Choose any highly symmetric geometric or algebraic object or shape (e.g. a polyhedron, a tiling of a plane, a knot, etc.) and either:

- try to "see" (or recognize) geometric configurations, which are already "hidden" in it, or
- "construct" (or draw) a configuration from this object in such a way that (some or all of) its symmetries are inherited by your configuration.

As a rule, many configurations (not just one) may be "found" in the same geometrical object or "constructed" from it.

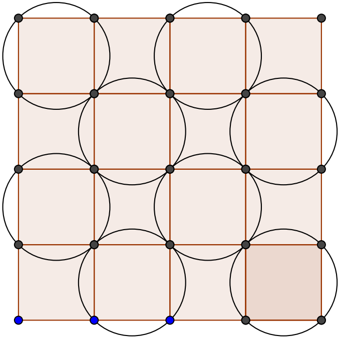
Example

Try to "see" in the figure below, besides the most obvious choice $(25_2, 10_5)$, also some other "hidden configurations" $\mathcal{C}$, such as (4_4) or (5_5) or (8_2) or (8_4) or (8_6) or (8_8) or $(16_4, 8_8)$.

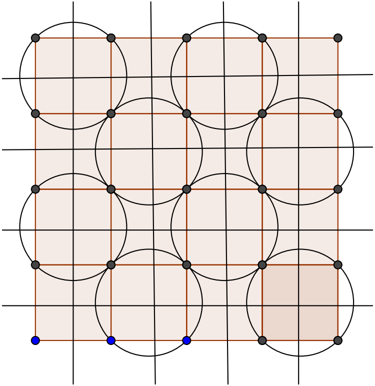


Hint: Just for a moment, forget the idea, that geometric configurations must contain only "points" and "lines".

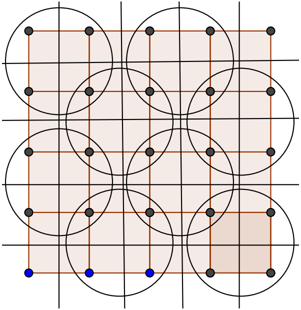
Trick 1. Be creative in interpreting "points" and "blocks" and even "incidences"!



Interpreting "points" as "circles around squares" and "blocks" as the remaining squares, and additionally regarding this geometric structure not as planar object but a tiling of the torus, we can now clearly "see" in it the "hidden" balanced configuration (8_4).



However, if we interpret "blocks" differently, as the "columns and lines" of a 4×4 matrix, this is then the configuration (8_2).



And if we interpret even "incidences" differently, as their exact opposite – "non-incidences", then we can "see" in this figure the configuration (8_6).

And the "chess knight interpretation" of incidences of 8 white and 8 black squares squares on the torus 4×4 gives us the configuration (8_8).

This is a good example of how our "seeing" depends on what we expect to see!

Configurations from geometric shapes

Many geometric representations of monovalent and balenced configurations may be obtained from the lists of

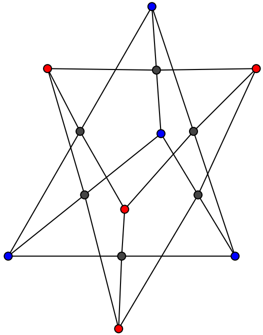
- uniform polyhedra
- compound bodies
- regular polytopes (and other "regular" shapes).

Example

The 12 vertices and the 12 pentagonal faces of the **great dodecahedron** (with five pentagons meeting at each vertex, intersecting each other making a pentagramic path) is a geometric representation of a configuration (12_5).

Example

The 8 vertices and the 8 edges of the **double tetrahedron** (whose edges intersect in their middle-points) represent points and lines of a geometric configuration (8_3);

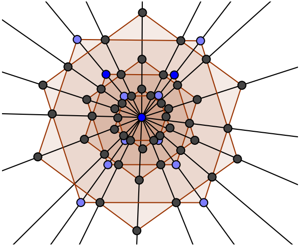


Smaller configurations as "stepping stones" to bigger configurations

Connecting "concentric" copies of C with "radial lines" from the centre raises the valences of the points by 1.

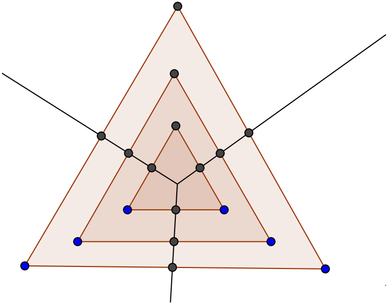
Example

Connecting 3 concentric copies of $(20_2, 10_4)$ by 20 radial lines we get a configuration $(60_3, 30_4)$.



Example

By the same "radial trick" we get from 3 copies of polivalent configuration $(3_2 3_1, 3_2)$, adding only 3 radial lines, a monovalent $(18_2, 9_3)$ configuration.

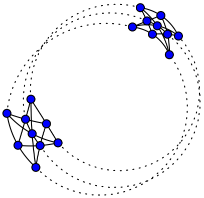


The same "radial trick" works for spatial configurations.

"Doubling" the valences of lines – a transformation $(n_k) \to (2n_k, n_{2k})$

Example

From two "antipodal" copies of the Pappus configuration (9_3) on a sphere we get a $(18_3, 9_6)$ configuration.

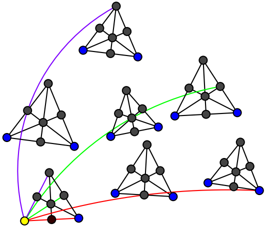


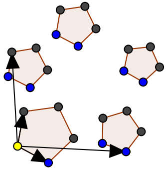
"Doubling" the valences of points

Place into each vertex of the configuration C its smaller copy C_i .
"Orient" all lines.
Each "directed edge" (u, v) in C has its "double" in C^2 connecting the corresponding vertices (u_i, v_j) in the copies C_i and C_j.

Example

Left: $(4_3 3_2, 6_3) \to (4_6 3_4, 12_3)$
Right: $(5_2) \to (5_4)$





Finding configurations may be turned into a game.

The "lattice points" of various grids (e.g. square, triangular, cubic grid) may help us in:

- finding examples of configurations with the given valence structure or symmetry;
- finding a geometrical realizations of a given combinatorial configuration.

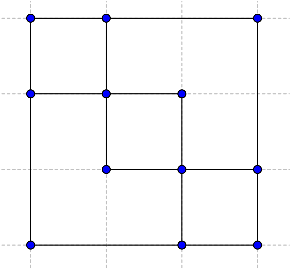
Method 2. Use planar and spatial grids and tesselations

To find a geometric configuration with a given valence structure and symmetry:

- place just a few "beans" on the points of chosen grid;
- gradually add more beans to get the right number of them;
- if necessary, remove or rearange some of them.

Example

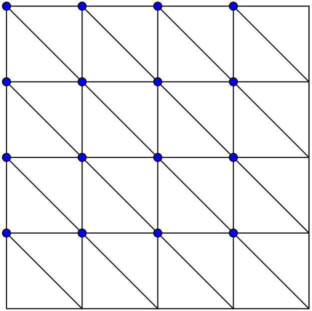
Here a planar square grid has been used to find an example of a $(12_2, 8_3)$ configuration.

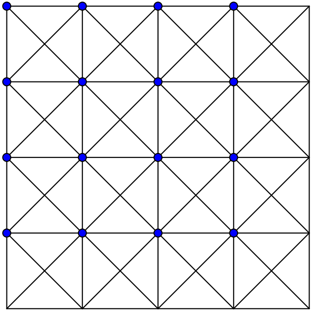


Configurations of points and lines on the torus (or other surfaces, tiled with squares, triangles or regular hexagons)

Example

Identifying the parallel sides of the planar square grid 5×5 we get a 4×4 grid on a torus which gives us a $(20_3, 15_4)$ configuration of points and lines, or a (20_4) configuration.



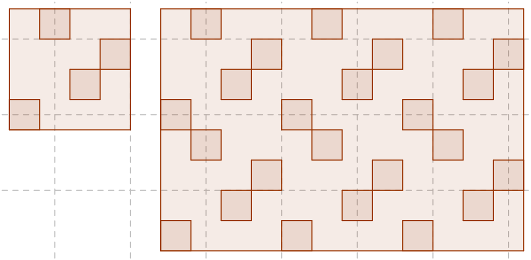


Permutation matrices $(k+1) \times (k+1)$ may be used as building blocks for obtaining $k-$configurations

Take $m \times n$ copies of a chosen $(k+1) \times (k+1)$ permutation matrix P.
Imagine this on a torus.
In each row and each column there are "paths" of length k composed of squares with entry 0.
Each horizontal path is incident with k vertical paths, and vice versa.
These "horizontal" and "vertical" paths will represent the "points" and "lines" of our configuration $C = (m \times n)_k$.

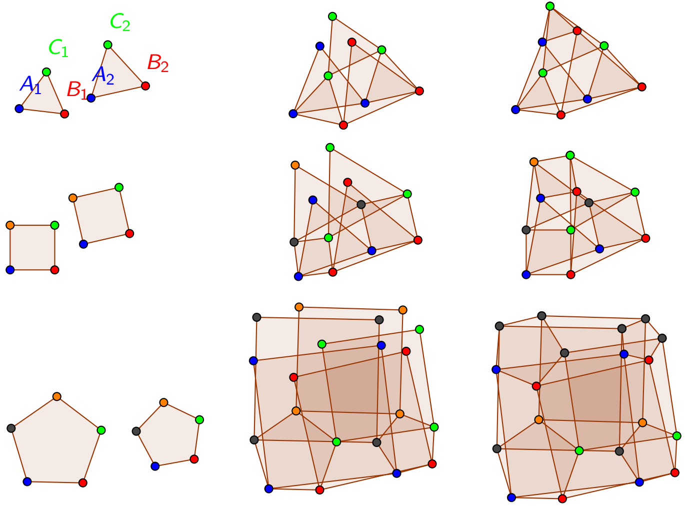
Example

A (24_3) configuration
of horizontal 3-paths ("points")
and vertical 3-paths ("blocks") on the torus
obtained from 3×2 copies of a 4×4 permutation matrix.



Another interpretation of the same figure on the torus gives us the configuration (24_4) whose "points" are "squares" and "blocks" are 3-paths of squares.

"Symmetric constructions" may produce symmetric "movable" configurations



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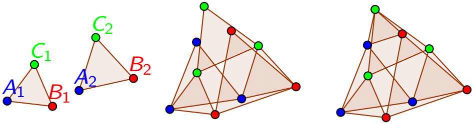
The following construction gives (for any integers $m, n \geq 3$) configurations (v_m, p_n) whose "points" and "blocks" are regular m-gons and n-gons:

- Start with any two (equally oriented) regular m-gons $(A_1, B_1, C_1, \ldots)$ and $(A_2, B_2, C_2, \ldots)$ in the plane.
- For each $i = 1, 2, \ldots, m$ construct regular n-gons $A_1, A_2, \ldots$.
- Then construct the n-gons $(A_j, B_j, C_j, \ldots)$ – they are all regular polygons (for every $j \in \{3, \ldots, n\}$).



The proof uses vectors and complex numbers: it is enough to show that in any of the constructed polygons the vector on each side B_jC_j is obtained by the rotation of the previous side A_jB_j for the same angle $\psi = 2\pi/n$.

<u>Proof.</u>

Let $\phi = 2\pi/m$ and $\psi = 2\pi/n$.
In the case $m = n = 3$ we start with two triangles, where we have the relations:
(1) $\mathbf{c}_1 - \mathbf{b}_1 = (\mathbf{b}_1 - \mathbf{a}_1)e^{i\phi}$
(2) $\mathbf{c}_2 - \mathbf{b}_2 = (\mathbf{b}_2 - \mathbf{a}_2)e^{i\phi}$
In the three "monochromatic" triangles we have the relations:
(4) $\mathbf{a}_3 - \mathbf{a}_2 = (\mathbf{a}_2 - \mathbf{a}_1)e^{i\psi}$
(5) $\mathbf{b}_3 - \mathbf{b}_2 = (\mathbf{b}_2 - \mathbf{b}_1)e^{i\psi}$
(6) $\mathbf{c}_3 - \mathbf{c}_2 = (\mathbf{c}_2 - \mathbf{c}_1)e^{i\psi}$
And we want to prove that the following relation holds:
(3) $\mathbf{c}_3 - \mathbf{b}_3 = (\mathbf{b}_3 - \mathbf{a}_3)e^{i\phi}$

The left side of (3) is:

$\mathbf{c}_3 - \mathbf{b}_3 =$

$(\mathbf{c}_2 + (\mathbf{c}_2 - \mathbf{c}_1)e^{i\psi}) - (\mathbf{b}_2 + (\mathbf{b}_2 - \mathbf{b}_1)e^{i\psi} =$

$(\mathbf{c}_2 - \mathbf{b}_2)e^{i\psi} + (\mathbf{c}_2 - \mathbf{c}_1)e^{i\psi}) - (\mathbf{b}_2 - \mathbf{b}_1)e^{i\psi} =$

$(\mathbf{b}_2 - \mathbf{a}_2)e^{i\phi}e^{i\psi} + ((\mathbf{c}_2 - \mathbf{b}_2) - (\mathbf{c}_1 - \mathbf{b}_1))e^{i\psi} =$

$(\mathbf{b}_2 - \mathbf{a}_2)e^{i\phi}e^{i\psi} + ((\mathbf{b}_2 - \mathbf{a}_2) - (\mathbf{b}_1 - \mathbf{a}_1))e^{i\psi}$

And the right side of (3) is the same:

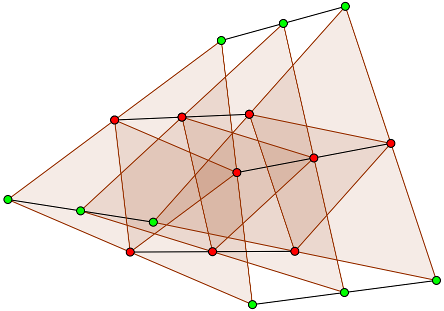
$(\mathbf{b}_3 - \mathbf{a}_3)e^{i\phi} =$

$(\mathbf{b}_2 + (\mathbf{b}_2 - \mathbf{b}_1)e^{i\psi})e^{i\phi} - (\mathbf{a}_2 + (\mathbf{a}_2 - \mathbf{a}_1)e^{i\psi})e^{i\phi}$

$(\mathbf{b}_2 - \mathbf{a}_2)e^{i\phi}e^{i\psi} + ((\mathbf{b}_2 - \mathbf{b}_1) - (\mathbf{a}_2 - \mathbf{a}_1)e^{i\psi}$

Example

A "movable" configuration of points and lines and equilateral triangles.

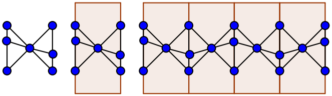


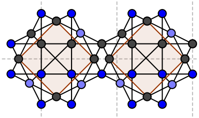
Modular and "mosaic" configurations

Method 3. Use "modules".

Modular configurations are constructed from the copies of a chosen set of "building blocks" – smaller configurations, which are "glued together" at some points or lines.

If these "building blocks" are inscribed in "mosaic pieces" and if we glue these pieces together along their boundary we get **"mosaic" configurations**, either planar, or spatial, or on some surface (e.g. on a torus).





Modules and (n_k) configurations

Modules and regular k- graphs help us find balanced k-configurations.

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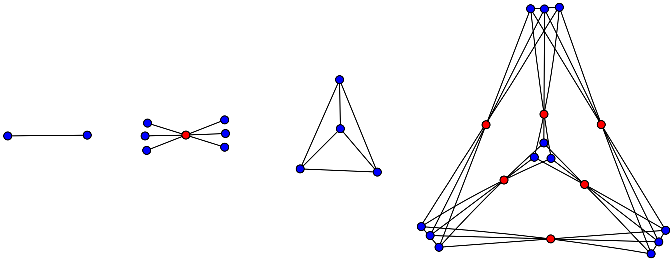
Take any 3-regular graph Γ on v vertices. Replace each of its $e = \frac{3v}{2}$ edges with a copy of configuration $(1_3 6_2, 7_3)$. You get a balanced (V_3, E_3) configuration C (where $V = 3v + e = E = 3e$), inheriting all the automorphisms of the graph Γ: $Aut(C) = Aut(\Gamma)$.

Example

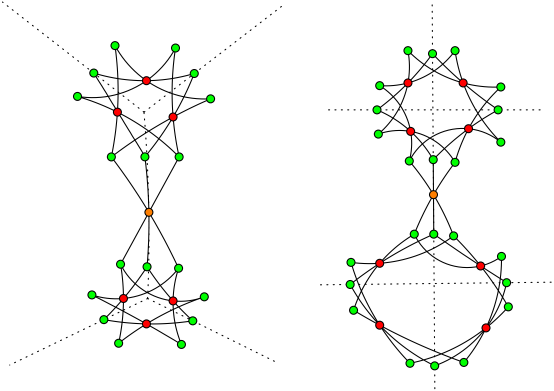
$(2_1, 1_2) \to (3_1, 6_2) \Rightarrow (4_3) \to (18_3)$.



A similar trick works on any (k-regular) graphs:

Example

Place "perpendicular" on each "arc" around each vertex 3 points and connect the "adjacent triples" of these points; then connect the adjacent triples on each edge. You get a balanced 3-configuration.



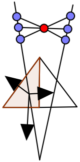
Corollary

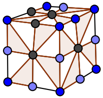
- If Γ is a "flag graph" of a polyhedron P, then we can construct a geometrical configuration $\mathcal{C} = (3e_3)$ which inherits all the symmetries of the polyhedron P:

$$Sym(\mathcal{C}) = Sym(P).$$

- If $\Gamma = Cay(G, S)$ is a Cayley graph of a group G with 3 generators in $S = \{g_1, g_2, g_3\}$, then the obtained $C = (3e_3)$ has all the automorphisms of the group G:

$$Aut(C) = G.$$



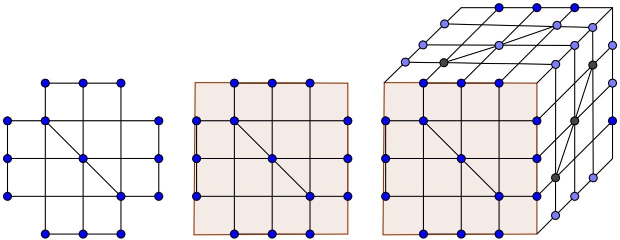


Mosaic polyhedral configurations

A mosaic spatial configurations has its "mosaic pieces" identified with the faces of some polyhedron.

Example

From 6 copies of the configuration $B = (3_{15}12_1,$
placed on the square "mosaic piece",
we may build a spatial "cubic" configuration $C = (45_3)$.



Platonic configurations

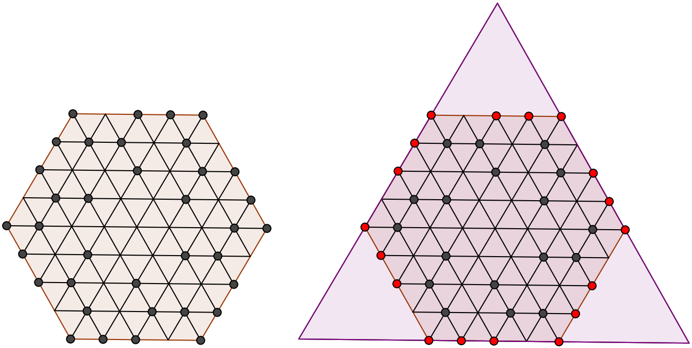
Platonic configuration is a geometric configuration $\mathcal{C}$ with symmetries of a Platonic solid $P \in \{T, C, O, D, I\}$:, or more precisely, such that $Sym(\mathcal{C}) \trianglelefteq Sym(P)$.

How to construct such configurations?

- The most obvious trick how to find examples of such configurations is by placing the copies of the same (suitable) planar configuration (with rotational or dihedral symmetry) on each of the faces of P; to get a connected mosaic configuration; some points must be placed on the edges or in the vertices of P.

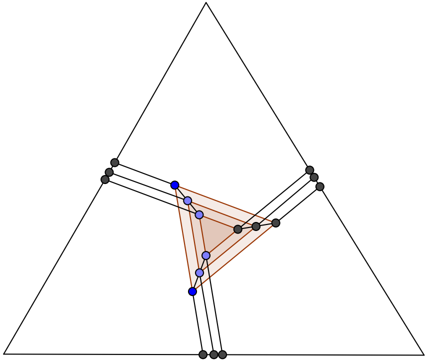
Example

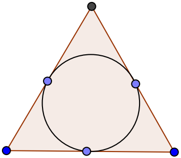
From the same planar configuration we get Platonic (n_4) configurations on T, O or I. Here the edges of triangular faces are not included among the lines of the configuration.



Example

Here the edges of P are included among the lines of Platonic (n_3) configurations on T, O and I.





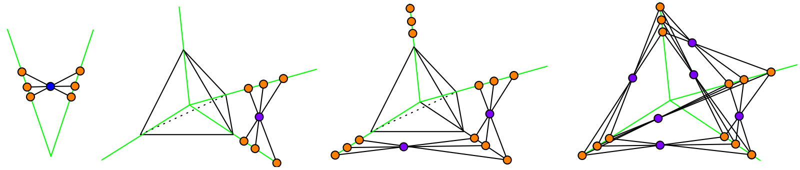
- In "mixed" configurations some blocks are represented by lines and some blocks are represented by circles.

"Spider-web" configurations

- Another useful trick is to place some points on the axes Ov, Oe, Of connecting the center O with vertices, midpoints of edges and midpoints of faces of P.

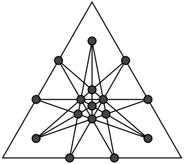
Example

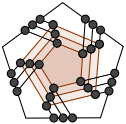
Using the 3-fold axes of tetrahedron as "linking places" we construct Platonic configuration (24_3).



Three types of Platonic configurations

Type 1: preserved by all *symmetries* of P (=rotations and reflections of E^3 preserving P)
Type 2: preserved only by (some or all) *rotational symmetries* of P.
Type 3: having only one reflection, or with a trivial symmetry group.
We are interested only in Types 1 and 2.



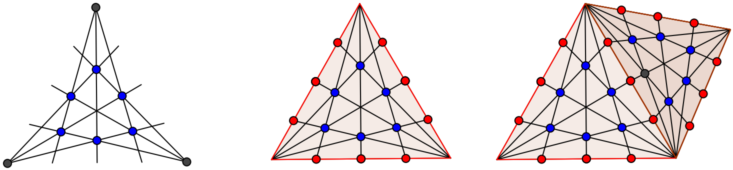


For any Platonic solid $P \in \{T, C, O, D, I\}$ let
P_k denote the class of all Platonic configurations (n_k) of Type 1 and let $P_{k,R}$ denote those of Type 2 which are preserved by all the rotational symmetries of P.

Many Platonic configurations may be obtained from the Pappus configuration (9_3).

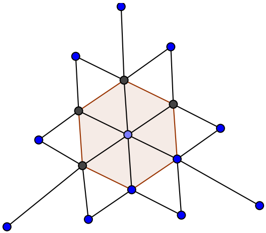
Example

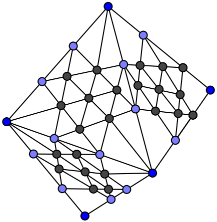
Replacing three 3-valent points with nine 1-valent points on the edges of the triangular faces we get Platonic (n_3) configurations on T, O and I.



Example

(n_4) configuration with the full symmetry group of the octahedron



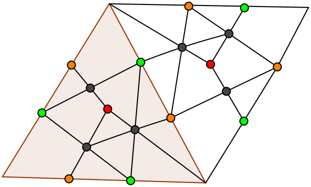


Another trick how to construct vertices of valence 3 on edges of P

Using different valences in edge points of the faces of P opens new possibilities in constructing Platonic configurations.

Example

A configuration with 3-fold rotational symmetry may have he edge points with valence 2 on one triangle and valence 1 on the adjacent triangle.

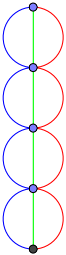


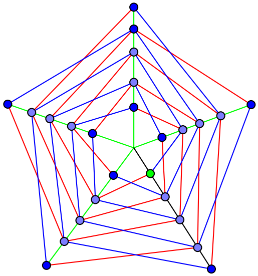
"Covering" configurations

Method 4. Use "voltages" on graphs or "configurations" to obtain "covering configurations"

Example

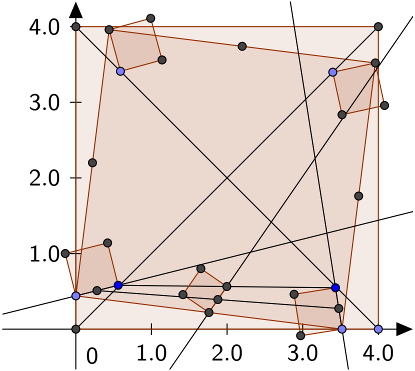
Using voltages 1 and -1 in $\mathbb{Z}_5$ we get the following "spiral" configuration $(25_3, 15_5)$.





Planar configurations with cyclical and dihedral symmetry

Evety planar configuration C may be used to obtain planar configuration with m-fold rotational or dihedral symmetry.



Other methods for constructing symmetric balanced configurations

These methods include the use of

- fundamental domains
- symmetrical polinomials in k variable
- k-regular graphs
- reduced Levi graphs
- voltage graphs producing "covering configurations"
- and many others.

Because of the "chameleon" nature of configurations there are many powerful tools for their study

Configurations may be constructed from or may be found in many different forms, as:

- bipartite graphs (Levi graphs), regular graphs
- lifts of reduced Levi graphs with given voltages
- incidence matrices
- algebraic expressions
- various geometric shapes (tilings, polyhedra, compounds, etc.)

All these various forms may be used (either independently or combined) as tools for constructions of **highly symmetric geometric configurations** (with many symmetries) or **highly symmetric combinatorial configurations** (with many automorphisms).

3. Open problems

Problem of existence

For which values of n and k
and for which Platonic solids P
and for which numbers p and q
exist Platonic configurations $P(p_q, n_k)$ (type 1)
or $P_R(p_q, n_k)$ (type 2)?

Problem of construction

And if they exist, how to find them?

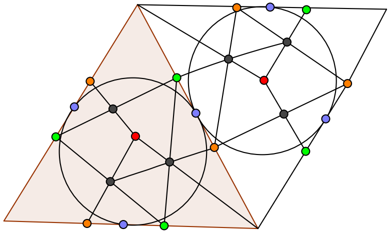
Problem of minimality

In particular, how to find *minimal examples* (with minimal number of points)?

Final words

Methods for constructing symmetric configurations are many.
The best results are obtained if we combine various methods.

Thank you!



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