Peano- and Hilbert curve

Historical comments

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Pilsen, Czech Republic

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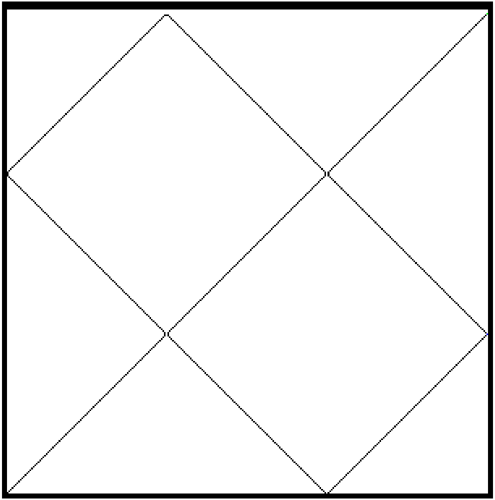
- Peano curve

Peano curve



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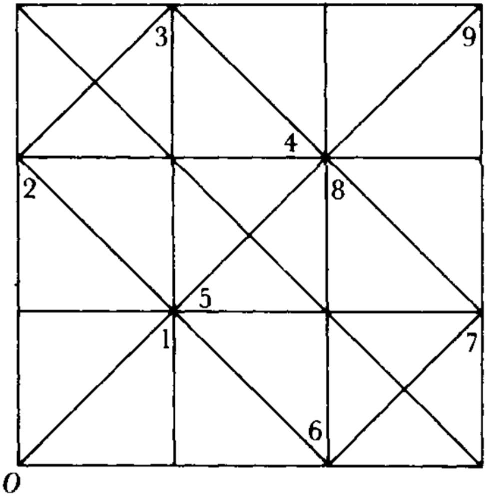

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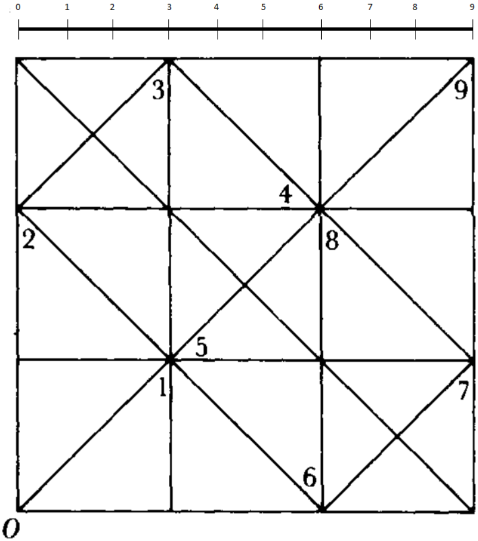

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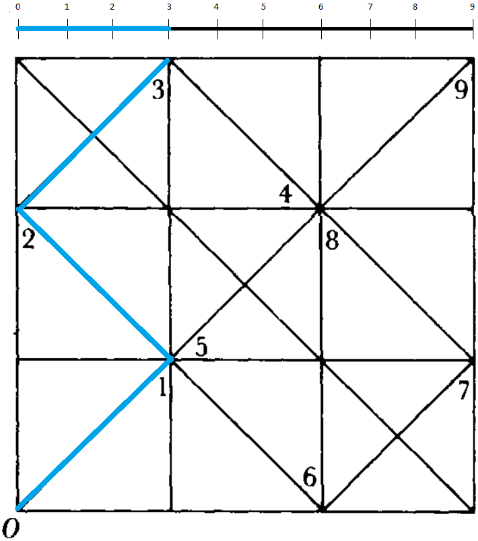

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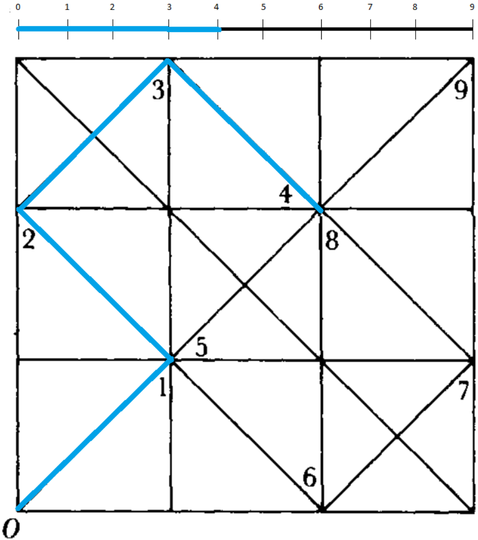

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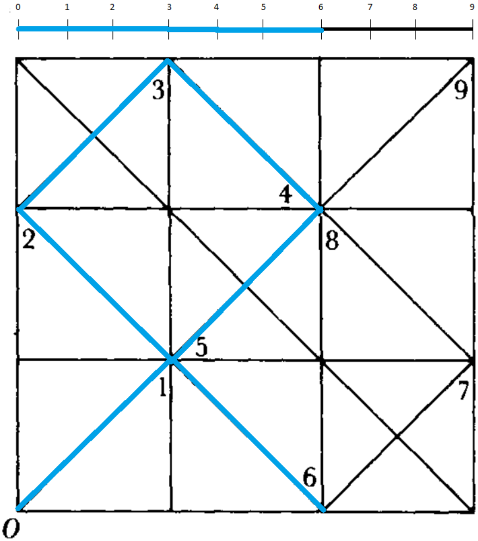

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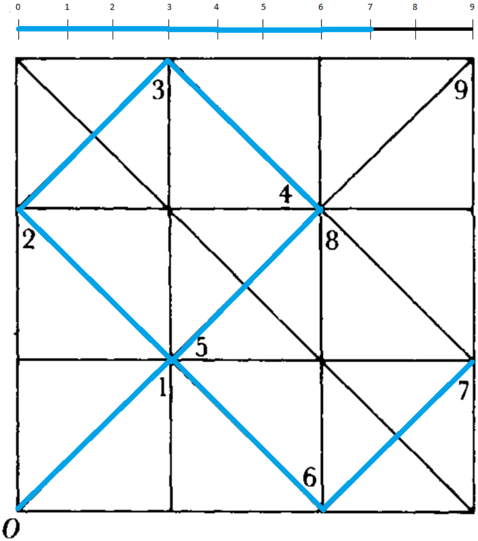

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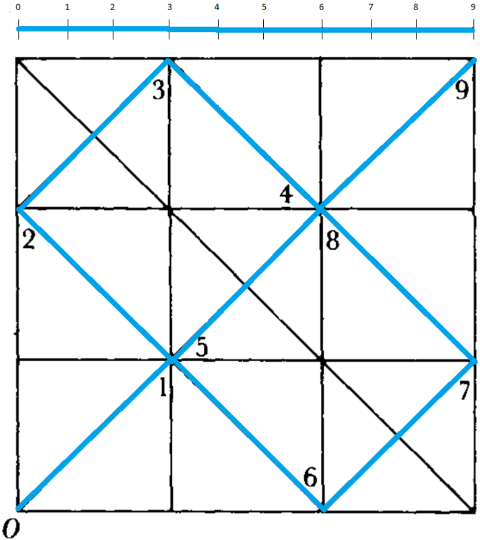

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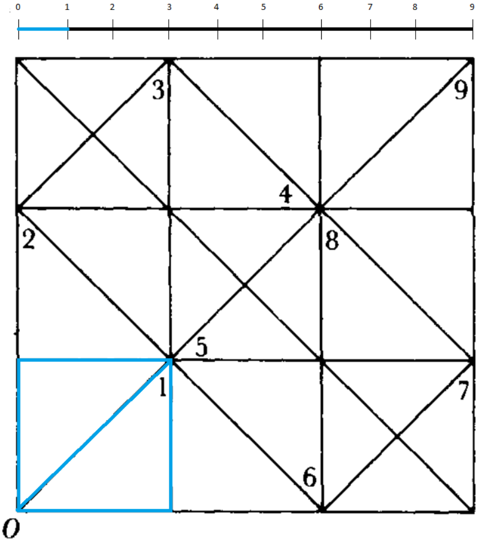

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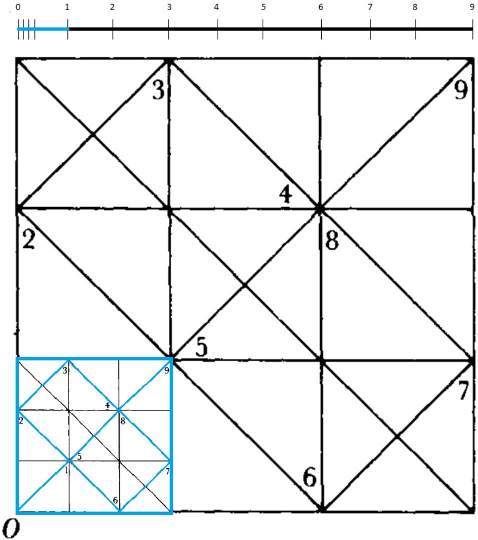

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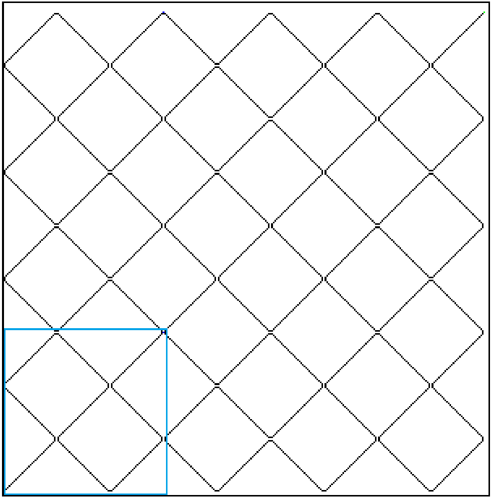

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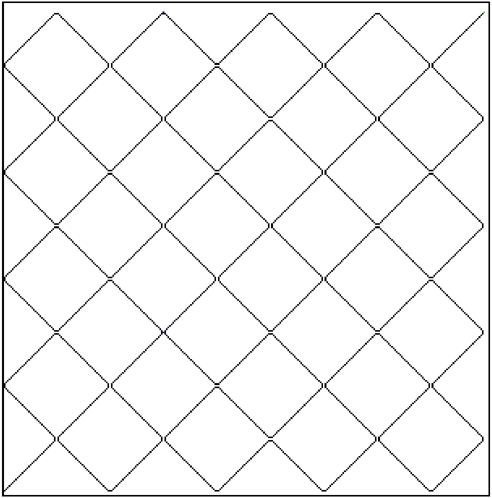

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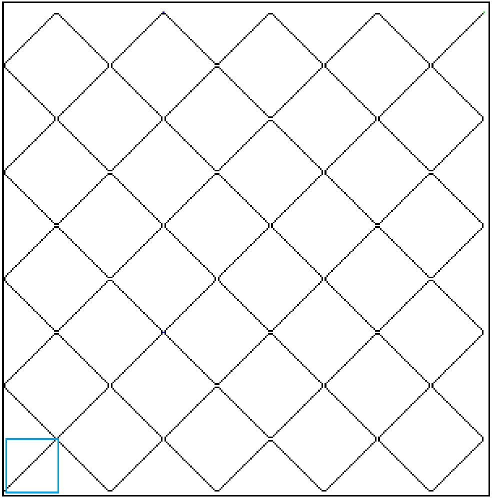
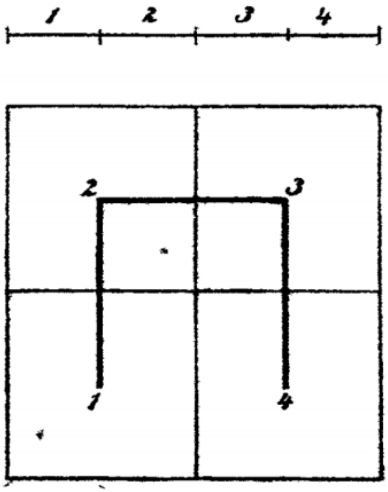


- Hilbert curve

Hilbert curve



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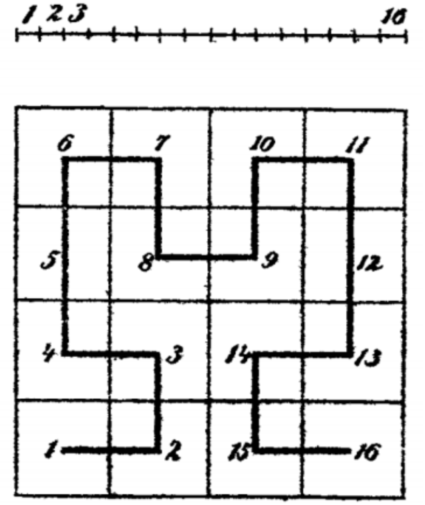

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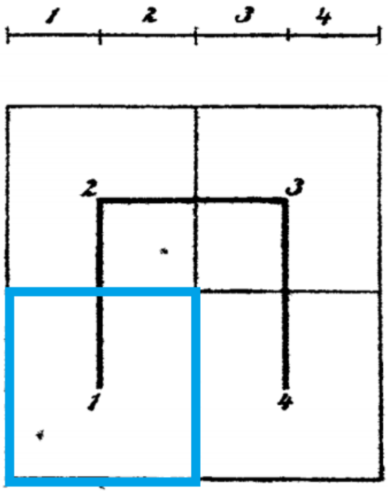

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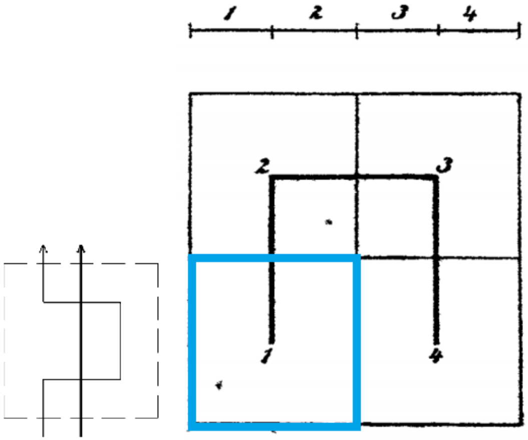

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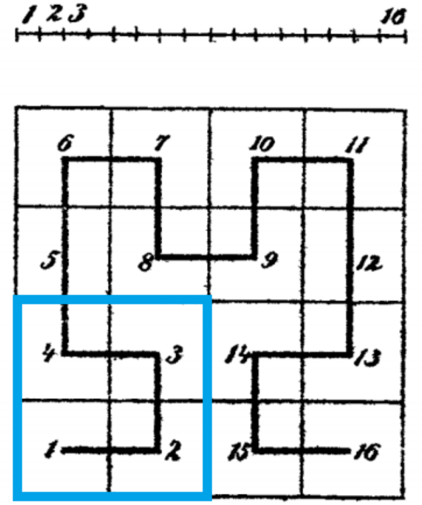

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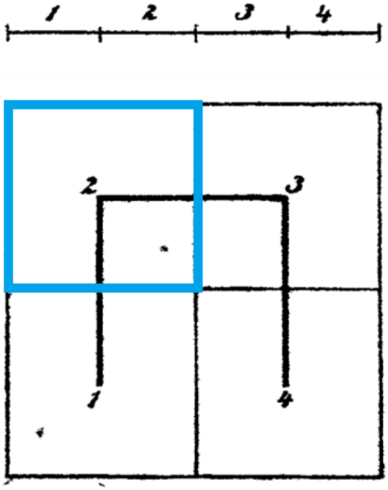

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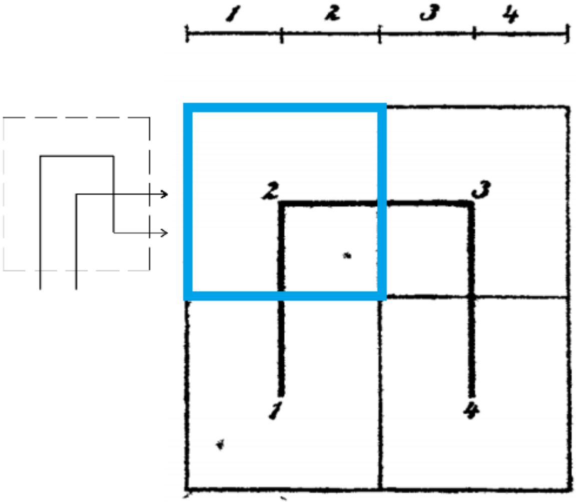

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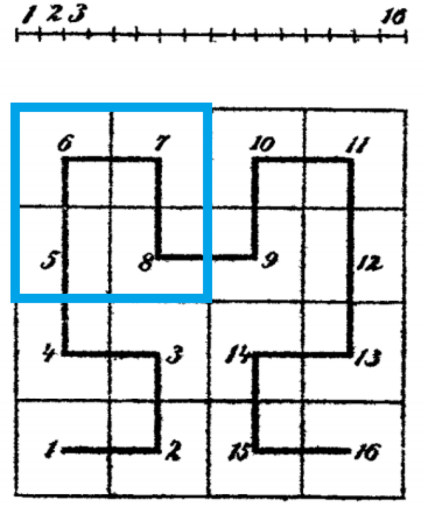

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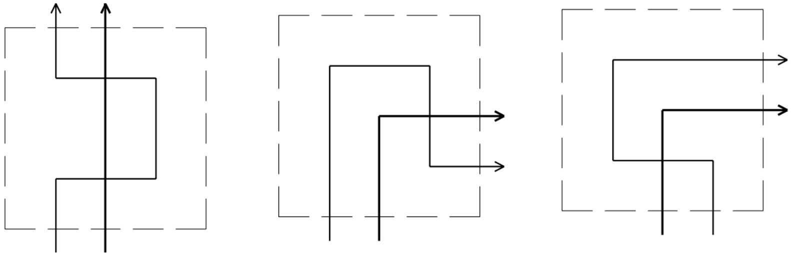

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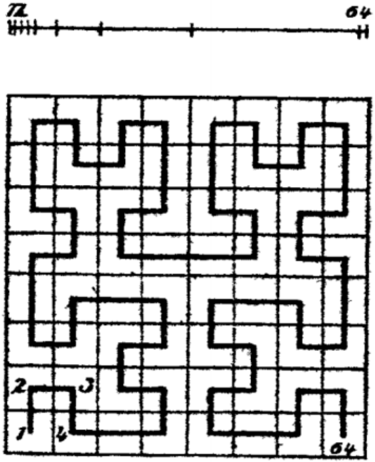
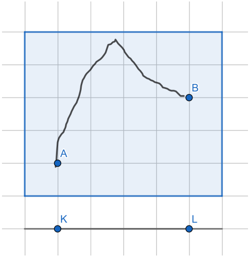


Applications

- excercise for recursion in the programming classes
- linearizing a discrete n-D space:
  - image rendering
  - indexing of n-D data in Geographic Information Systems,
  - scheduling of the multimedia server requests

Peano and Hilbert curve – some facts

- mapping functions are piecewise linear
- everywhere continuous, nowhere differentiable
- So Hilbert: "A point in the move can go through all the points of the square in the **finite** time."
- **not a bijection** (only continuous, cannot be also one-to-one):



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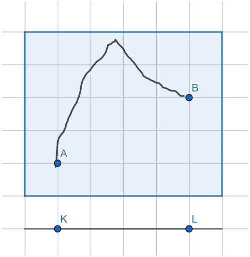
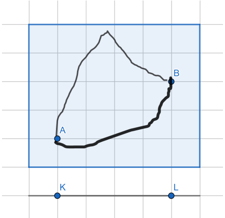
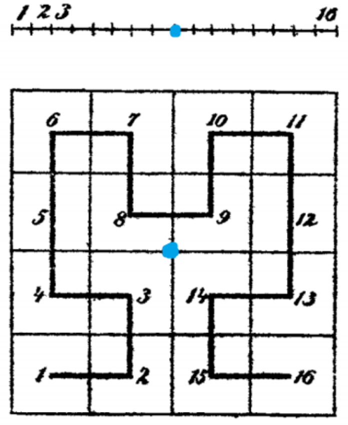




- So Hilbert: “The **inverse mapping** assigns to each point of the square 1, 2 or 4 points on the line.”

- Minkowski to Hilbert 22.12.1890: “Are you sure that when you move a point in the square, the point passes some places at three different times? It seems to me that it does not go anywhere more than twice.”
- Hilbert was right– 1x, 2x, 4x but also 3x (proof Sierpinski 1912)

Hilbert curve



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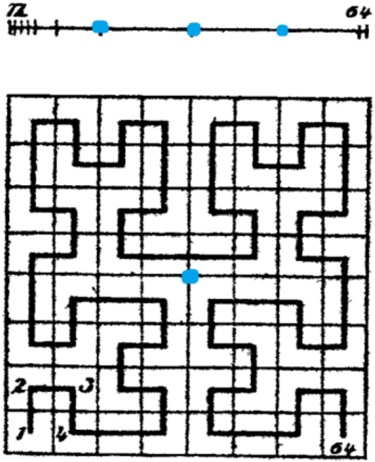

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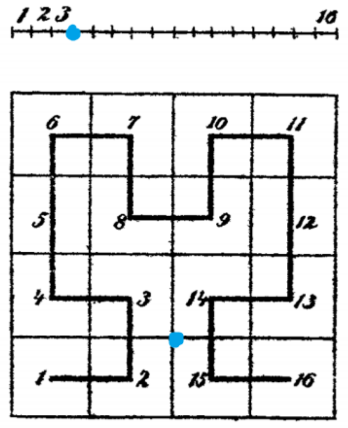

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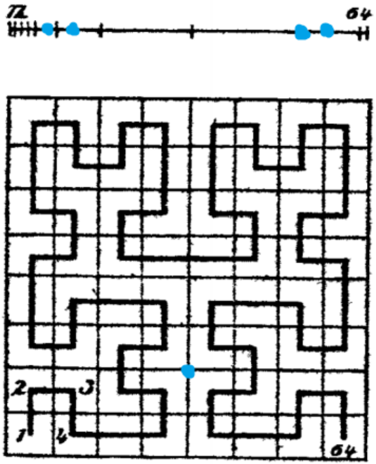


Historical comments

Giuseppe Peano (1858–1932)

In Turin: 1876–1932

- 1889 Arithmetices principia [Peano axioms]
- 1890 **Sur une courbe, qui remplit toute une aire plane** [Peano curve]
- 1895–1908 Formulario matematico [Symbolical logic]

David Hilbert (1862–1943)

In Königsberg: 1886–1895,
In Göttingen: 1895–1943

- 1891 **Über die stetige Abbildung einer Linie auf ein Flächenstück** [Hilbert curve]
- 1900 Mathematische Probleme [Hilbert's problems]
- 1931 Grundlagen der Mathematik [Formal systems (with P. Bernays)]

We must know
We shall know

Hilbert's relation to Peano

- no Hilbert-Peano letters
- no mention of Peano in Hilbert's colloquium diary from the concerned period
  (although Hilbert recommended Peano's works later)
- Hilbert referred to Peano only in this paper on Hilbert curve, not before and after that

Hilbert's relation to Peano – Hilbert's Problems

- 2nd International Congress of Mathematicians in Paris 1900
- Hilbert's 23 mathematical problems for the 20th century
- Peano in the audience
  2nd problem - consistency of arithmetics
  Peano objected that Hilbert ommited results of Italians
- Hilbert did not revise his text by this 2nd problem
- so why Hilbert curve?

Minkowski's letter to Hilbert 22.12.1890

I recently once thought about your presentation at the meeting of the scientists [. . .] What do you have against the in principle simpler example, that the time from 0 to 1 is continuously represented as a decimal number and from the **even and odd digits** alone, two other decimal numbers are formed that should just express the **right-angled coordinates** of the point in the corresponing time. The continuity of movement is just preserved here in exactly the same sense.

Mapping by decimal development (1)

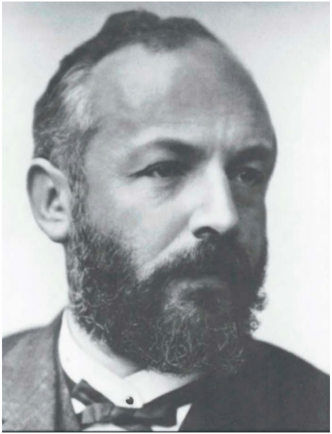
- $\frac{\sqrt{2}}{2} = 0,\underline{7}0\underline{7}1\underline{0}6\underline{7}8\ldots$
  $\Downarrow$
  $\mathbf{x = 0{,}7707\ldots}$

Mapping by decimal development (2)

- $\frac{\sqrt{2}}{2} = 0,7\underline{0}7\underline{1}0\underline{6}7\underline{8}\ldots$
  $\Downarrow$
  $x = 0,7707\ldots$
  y = 0,0168...

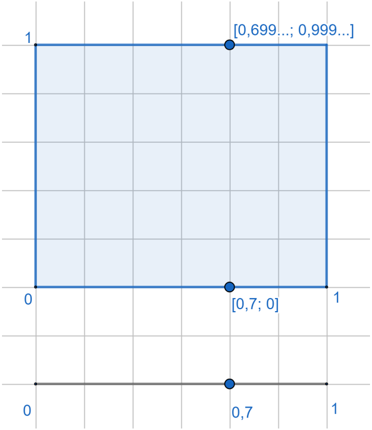
Completed history – Georg Cantor (1845–1918)

- 1877 1-D to n-D mapping by **decimal developments**
- 1882 Continuum hypothesis: *every subset of real numbers can be bijectively mapped either to the set of natural numbers or to the set of real numbers.*



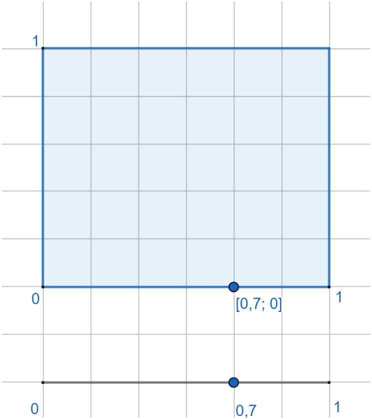
Mapping by decimal development (3)

- Problem:
  $0,7 = 0,6\overline{9}$
  $\Downarrow$
- $0,6\underline{9}9\underline{9}9\underline{9}\ldots$
  $x = 0,6999\ldots$
  $\mathbf{y = 0{,}9999\ldots}$
- $0,7\underline{0}0\underline{0}0\underline{0}\ldots$
  $x = 0,7000\ldots$
  $\mathbf{y = 0{,}0000\ldots}$



Mapping by decimal development (4)

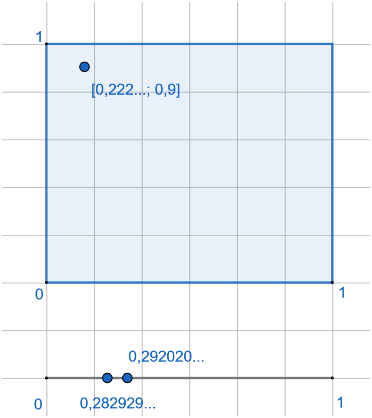
- Problem:
  $0,7 = 0,6\overline{9}$
- ~~$0,6\underline{9}9\underline{9}9\underline{9}\ldots$~~
  ~~$x = 0,6999\ldots$~~
  ~~**y = 0,9999...**~~
- $0,7\underline{0}0\underline{0}0\underline{0}\ldots$
  $x = 0,7000\ldots$
  y = 0,0000...



Mapping by decimal development (5)

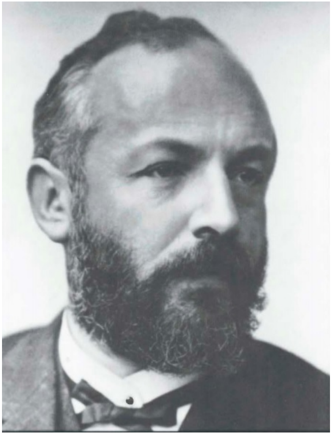
- Problem:
  $0,28\overline{29}$
  $\Downarrow$
  $x = 0,22222\ldots$
  $y = \sout{0,89999\ldots} = 0.9$

- but $0,29\overline{20}$
  $\Downarrow$
  $x = 0,22222\ldots$
  $y = 0.9$



Completed history – Georg Cantor (1845–1918)

- 1877 1-D to n-D mapping decimal developments
- 1878 1-D to n-D mapping by **continuous fractions**
- 1882 Continuum hypothesis: *every subset of real numbers can be bijectively mapped either to the set of natural numbers or to the set of real numbers.*



Mapping by continuous fractions - example 1

$$\sqrt{2} - 1 = [2, 2, 2, 2, 2, \ldots] =$$

$$= \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}}$$

Mapping by continuous fractions - example 1

$$\sqrt{2} - 1 = [\underline{2},\ 2,\ \underline{2},\ 2,\ \underline{2},\ \ldots] =$$

$$= \cfrac{1}{\underline{2} + \cfrac{1}{2 + \cfrac{1}{\underline{2} + \cfrac{1}{2 + \cfrac{1}{\underline{2} + \cdots}}}}}$$

$$\Downarrow$$

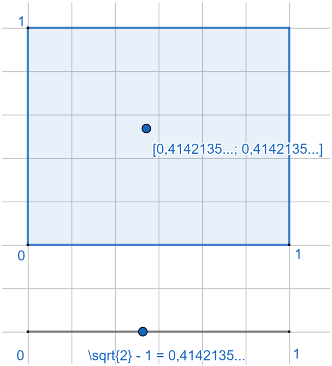
$$x = [2, 2, 2, \ldots] = \sqrt{2} - 1$$

Mapping by continuous fractions - example 1

$$\sqrt{2}-1 = [2,\ \underline{2},\ 2,\ \underline{2},\ 2,\ \ldots] = \cfrac{1}{2+\cfrac{1}{\underline{2}+\cfrac{1}{2+\cfrac{1}{\underline{2}+\cfrac{1}{2+\cdots}}}}}$$

$$\Downarrow$$

$$x = [2,2,2,\ldots] = \sqrt{2}-1$$
$$y = [2,2,2,\ldots] = \sqrt{2}-1$$



Mapping by continuous fractions - example 2

$$\frac{3}{2} - \sqrt{2} = [11, \overline{1, 1, 1, 10}] =$$
$$= [\underline{11}, 1, \underline{1}, 1, \underline{10}, 1, \underline{1}, 1, \underline{10}, \ldots]$$

$\Downarrow$

$$x = [11, \overline{1, 10}] = 6 - \sqrt{35}$$

Mapping by continuous fractions - example 2

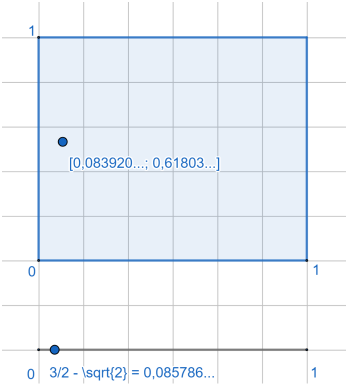
$$\frac{3}{2} - \sqrt{2} = [11, \overline{1, 1, 1, 10}] =$$

$$= [\mathbf{11},\ \underline{\mathbf{1}},\ \mathbf{1},\ \underline{\mathbf{1}},\ \mathbf{10},\ \underline{\mathbf{1}},\ \ldots]$$

$$\Downarrow$$

$$x = [11, \overline{1, 10}] = 6 - \sqrt{35}$$

$$y = [1, 1, 1, \ldots] = \tfrac{\sqrt{5}-1}{2}$$



Reason for Hilbert's interest

	1:1 mapping	continuous
Cantor's proof	x	
Peano- and Hilbert curve		x

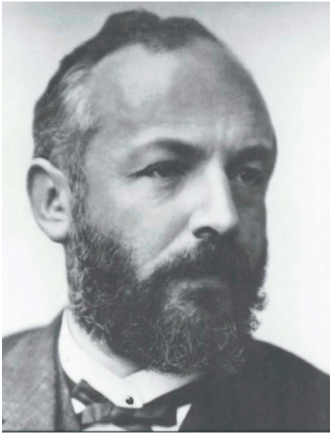
Completed history – Karl Weierstrass (1815–1897)

- Hilbert referred only to Peano and Weierstrass:
- Weierstrass's approximation theorem:

  every continuous function can be approximated *by the limit of the sequence of polynomial functions, uniformly convergent on the whole interval*
- Hilbert curve is continuous $\Rightarrow$ the analytic expression can be given

Completed history – Georg Cantor (1845–1918)

- 1877 1-D to n-D mapping by decimal developments
- 1878 1-D to n-D mapping by continuous fractions
- 1882 Continuum hypothesis
- —
- 1890 Peano mentioned Cantor's proof by continuous fractions
- 1891 **Hilbert did not mention Cantor**



Hilbert's reasons for Hilbert curve

- More intuitive version of Peano's continuous mapping?

Hypothesis: Hilbert's affinity to Cantor's set theory

references:

- 1891 Hilbert curve

  $\Downarrow$

-
-
-
-
-

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  "No one shall us expell from the paradise that Cantor created for us."
- 1930 Hilbert's biography

Conclusion

- Hilbert did not find much interest in the research of Peano
- Hilbert created Hilbert curve to support Cantor

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