Univerza *v Ljubljani*

Bonded knots

A topological model for knotted proteins

Boštjan Gabrovšek

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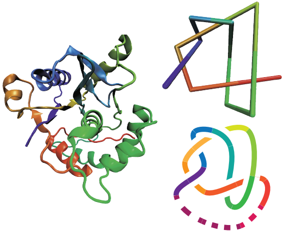
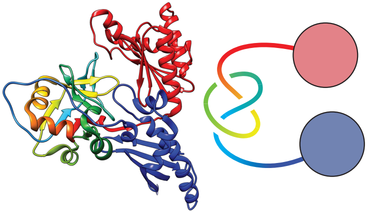
History of protein knots

- 1994: existence of knotted proteins proposed (Mansfield)
- 1994: first knotted protein found (Liang, Mislow)
- 2000: first deep knot found, 3_1 in 4_1 (Taylor)
- 2014: knotted protein database knotprot.cent.uw.edu.pl



Protein UCHL3 contains the knot 5_2



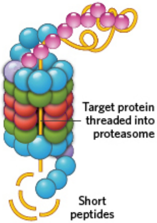
Protein Tp0642, deepest knot found up to date (Lim, Jackson, 2015)

Questions/Problems

- Why are proteins knotted (evolutionary advantages)?
- How do protein form knots in microbiological processes?
- How do we distinguish/classify/analyse such structures?

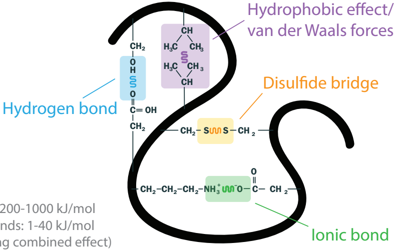
Hypothesised (biological) advantages of knotted proteins:

- increases thermal stability
- increases kinetic stability
- increases chemical stability
- prevention to being pulled into the proteasome



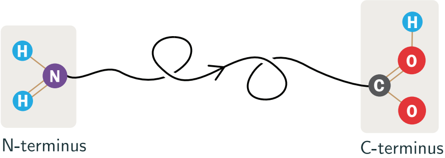
Bonds & Orientation

The three-dimensional protein structure also consists of *bonds* tying parts of the peptide backbone. These bonds have both a structural and functional role and can be of several types.



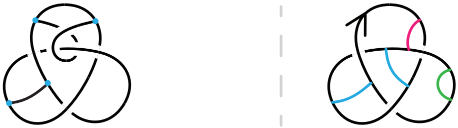
covalent bonds: 200-1000 kJ/mol
non-covalent bonds: 1-40 kJ/mol
(can have a strong combined effect)

The protein backbone also has a natural orientation



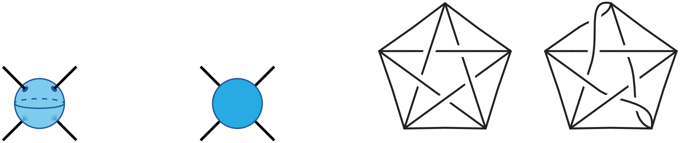
Spatial graphs

We can model *a protein with bonds* as:



3-valent spatial graph — bonded knot

We distinguish between *non-rigid* graphs and *rigid* graphs.



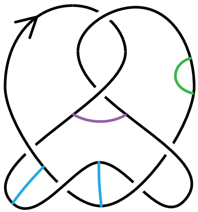
non-rigid vertex — rigid vertex — inequivant rigid-vertex spatial graphs

Rigid bonded knots are easier to study, but *non-rigid knots* better reflect spatial isotopy.

Non-rigid bonded knots (G., 2019)

A *(non-rigid) colored bonded knot* is the triple $(K, \mathcal{B}, c)$, where:

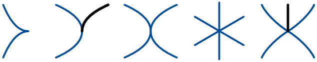
- $K \hookrightarrow \mathbb{R}^3$ is an oriented *knot*,
- $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ is the set of bonds neatly embedded into $(\mathbb{R}^3, K)$,
- $c : \mathcal{B} \to \mathbb{N}$ is the *coloring function*.



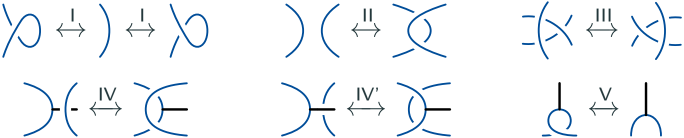
Two bonded knots are *equivalent* if they are ambient isotopic.

Reidemeister moves

A *diagram* of a bonded knots K is a *regular projection* of K to a plane.

Forbidden positions: 

Reidemeister moves:



Theorem

Two non-rigid bonded knot diagrams represent isotopic knots iff they are connected though a finite sequence of moves I–V.

In order to study *rigid isotopy*, we replace move V by

Rigid bonded knots

Let $\mathcal{D}$ be the set of all colored bonded knot diagrams.

Rigid (colored) bonded knots are equivalence classes

$$\bar{\mathcal{L}} = \mathcal{D}/\sim,$$

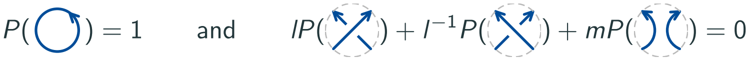
where $D_1 \sim D_2$ iff thy are connected through planar isotopy and a finite sequence of moves I–IV and RV.

The HOMFLYPT polynomial

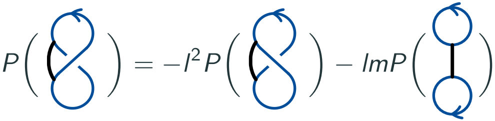
The HOMFLYPT polynomial of classical knots

$$P : \mathcal{L}(S^3) \to \mathbb{Z}[l^{\pm 1}, z^{\pm 1}]$$

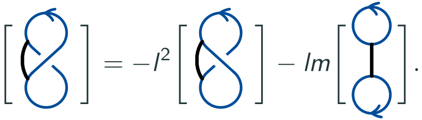
is defined using skein relations:



In the case of bonded knots P is not well defined:



We will form am R-module in which it holds



The HOMFLYPT skein module of bonded knots

Let

- $\mathcal{L}$ be the set of all non-rigid bonded links,
- R be a commitative ring with units l in m (also let $l^2 + 1$ and $l^2 \pm ml + 1$ be invertible in R),
- $R[\mathcal{L}]$ be the free R-modul generated by $\mathcal{L}$,
- $S(R, l, m)$ be the submodule generated by expressions

$l \Big[$  $\Big]_{\mathcal{B}} + l^{-1} \Big[$  $\Big]_{\mathcal{B}} + m \Big[$  $\Big]_{\mathcal{B}}$

The *non-rigid HOMFLYPT skein module* is the quotient module

$$\mathcal{H}(R, l, m) = R[\mathcal{L}]/S(R, l, m)$$

By taking $\bar{\mathcal{L}}$ to be the set of rigid bonded knots, we similarly define the *rigid HOMFLYPT skein module* $\bar{\mathcal{H}}(R, l, m)$

The HOMFLYPT skein module

We define the following *elementary bonded knots* with color i:

$\Theta_i =$ 

, $\bar{\Theta}_i = i$ 
, $H_i = i$ 
$\bar{H}_i = i$ 
,

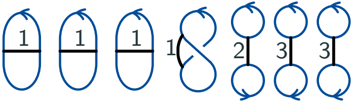
Theorems (G., 2020)

1. The H.S.M. of *rigid* bonded knots $\bar{\mathcal{H}}$ is freely generated by

$$\bar{\mathcal{B}} = \left\{ \prod_{i=1}^{k} \Theta_i^{m_i} \bar{\Theta}_i^{\bar{m}'_i} H_i^{n_i} \bar{H}_i^{n'_i} \mid k \in \mathbb{N};\ \vec{m}, \vec{m}', \vec{n}, \vec{n}' \in \mathbb{N}_0^k \setminus \vec{0} \right\} \cup \{U\}.$$

2. The H.S.M. of *non-rigid* bonded knots $\mathcal{H}$ is freely generated by

$$\mathcal{B} = \left\{ \prod_{i=1}^{k} \Theta_i^{n_i} \mid \vec{n} \in \mathbb{N}_0^k \setminus \vec{0} \right\} \cup \{U\}.$$

E.g. $\Theta_1^3 \bar{\Theta}_1 H_2 \bar{H}_3^2 =$ 

Idea of proof (generating set)

First, we show that $\mathcal{B}$ is the generating set taking these steps:

1. isolate the bond,
2. show that this bond can be "cut out" and expressed as a linear combination of knots and Θ's and H's,
3. repeat the process until no bonds left.

Using the HOMFLYPT relation, we can compute:

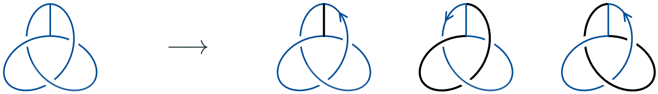
$(l^2 + lm + 1)(l^2 - lm + 1)$  $= l^2 m^2 \Big($  $\cdot H_i +$  $\cdot \Theta_i \Big) + \frac{l^3 m^3}{1+l^2} \Big($  $\cdot \Theta_i +$  $\cdot H_i \Big).$

and

$(l^2 + lm + 1)(l^2 - lm + 1)$  $= l^2 m^2 \Big($  $\cdot \bar{H}_i +$  $\cdot \bar{\Theta}_i \Big) + \frac{l^3 m^3}{1+l^2} \Big($  $\cdot \bar{\Theta}_i +$  $\cdot \bar{H}_i \Big).$

Example (non-rigid case)

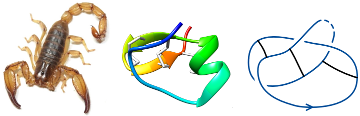
We can associate three bonded knots to the theta-curve $\Theta 3_1$.



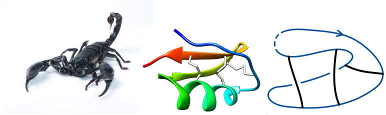
$\Big[$  $\Big]_{\mathcal{B}} = (l^{-2}m^{2} - 2l^{-2} - l^{-4})\,\Theta$

$\Big[$  $\Big]_{\mathcal{B}} = \Theta$

Example (rigid case)



CN29 toxin (Mexican Nayarit Scorpion)



ADWX-1 toxin (Chinese scorpion)

$\left[K_{\text{CN29}}\right]_{\bar{\mathcal{B}}} = \frac{1}{(1+l^2)^2(l^2+ml+1)^2(l^2-ml+1)^2}\Big(l^6m^4(-1-3l^2-3l^4-l^6+l^2m^2+2l^4m^2)$ 

$+\, l^5m^3(1+3l^2+3l^4+l^6-m^2-6l^2m^2-6l^4m^2-l^6m^2+l^2m^4+3l^4m^4)$ 

$+\, l^7m^5(-1-l^2+l^2m^2)$  $+\, l^6m^6(-1-2l^2+l^2m^2)$ 

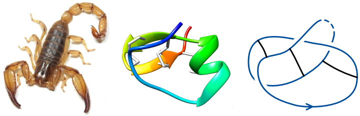
$+\, l^6m^4(-1-2l^2-l^4-m^2-l^2m^2+l^4m^2+l^2m^4)$  $+\, l^5m^5(-1-3l^2-2l^4+l^2m^2+l^4m^2)$  $\Big)$

$\left[K_{\text{ADWX-1}}\right]_{\bar{\mathcal{B}}} = \frac{1}{(1+l^2)^2(l^2+ml+1)^2(l^2-ml+1)^2}\Big(l^6m^4(-1-2l^2-l^4+l^4m^2)$ 

$+\, l^7m^5(-4-4l^2+2l^2m^2)$  $+\, l^7m^5(-1+l^4)$  $+\, l^7m^5(-2-2l^2+l^2m^2)$ 

$+\, l^4m^4(1+2l^2+l^4-2l^2m^2-3l^4m^2+l^4m^4)$  $+\, l^6m^4(-2-4l^2-2l^4+2l^4m^2)$  $\Big)$

Example (rigid case)



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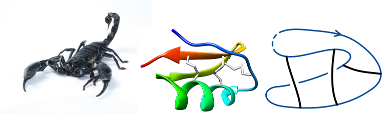

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$+\, l^6m^4(-1-2l^2-l^4-m^2-l^2m^2+l^4m^2+l^2m^4)$  $+\, l^5m^5(-1-3l^2-2l^4+l^2m^2+l^4m^2)$  $\Big)$

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$+\, l^7m^5(-4-4l^2+2l^2m^2)$  $+\, l^7m^5(-1+l^4)$  $+\, l^7m^5(-2-2l^2+l^2m^2)$ 

$+\, l^4m^4(1+2l^2+l^4-2l^2m^2-3l^4m^2+l^4m^4)$  $+\, l^6m^4(-2-4l^2-2l^4+2l^4m^2)$  $\Big)$

Thank you!