Barycentric Configurations in Real Space

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My main interest in (realisable) configurations stems from the theory of Tits-buildings

"Which finite buildings
can we realise
in Euclidean space?"

Point-line buildings: generalised polygons

General observations:

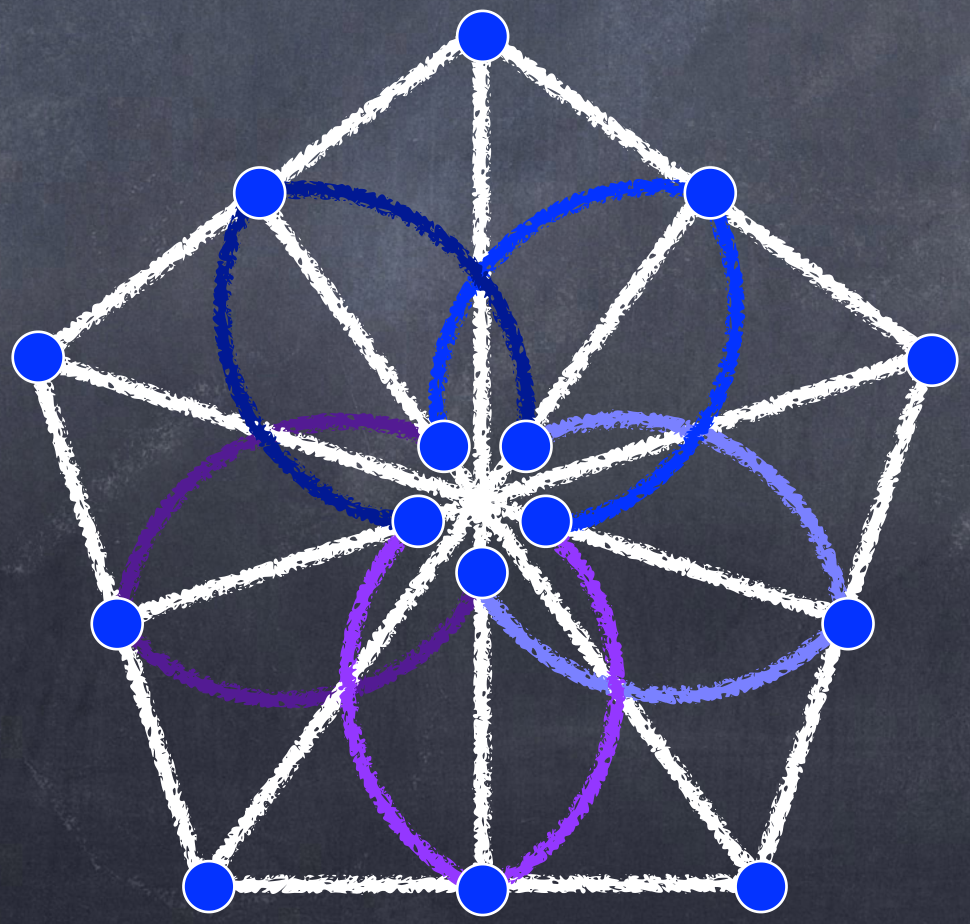
- only small examples are realisable
- small examples have large symmetry groups

Example

The smallest generalised quadrangle

Point set = {pairs from fixed 6-set {1,2,3,4,5,6}}
Line set = {partitions of {1,2,3,4,5,6} in pairs}

Doily



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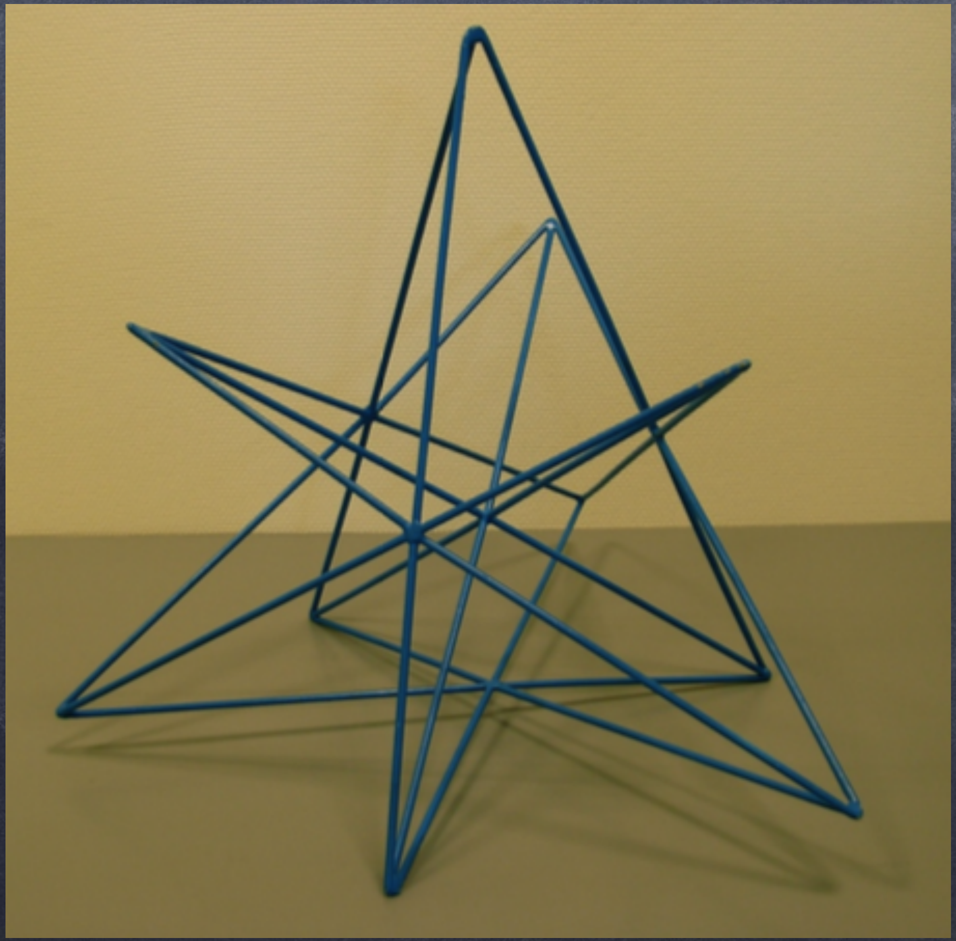

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Theoretic explicit construction of realisation (over any field k):

In hyperplane $x_1+x_2+x_3+x_4+x_5+x_6=0$ of PG(5,k) take for {i,j} the point $x_i=x_j=-2$, $x_k=1$ for $k\neq i,j$

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Example of a line: (-2,-2,1,1,1,1)
(1,1,-2,-2,1,1)
(1,1,1,1,-2,-2)

Sum of the coordinates = (0,0,0,0,0,0)!

Sum of the coordinates of the points on a line = (0,0,0,0,0,0)

→ Barycentric realisation

→ Barycentric configuration

Let us construct this barycentric representation in another way

Consider a symmetric incidence matrix A

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	{1,2}
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	{3,4}
1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	{5,6}
0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	{3,5}
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	{4,6}
0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	{3,6}
0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	{4,5}
0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	{1,5}
0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	{2,6}
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	{1,6}
0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	{2,5}
0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	{1,3}
0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	{2,4}
0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	{1,4}
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	{2,3}

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	{1,2}	-2
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	{3,4}	1
1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	{5,6}	1
0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	{3,5}	1
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	{4,6}	1
0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	{3,6}	1
0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	{4,5}	1
0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	{1,5}	-2
0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	{2,6}	1
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	{1,6}	-2
0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	{2,5}	1
0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	{1,3}	-2
0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	{2,4}	1
0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	{1,4}	-2
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	{2,3}	1

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	{1,2}	−2
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	{3,4}	1
1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	{5,6}	1
0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	{3,5}	1
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	{4,6}	1
0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	{3,6}	1
0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	{4,5}	1
0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	{1,5}	−2
0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	{2,6}	1
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	{1,6}	−2
0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	{2,5}	1
0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	{1,3}	−2
0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	{2,4}	1
0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	{1,4}	−2
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	{2,3}	1

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	{1,2}	–2
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	{3,4}	1
1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	{5,6}	1
0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	{3,5}	1
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	{4,6}	1
0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	{3,6}	1
0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	{4,5}	1
0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	{1,5}	–2
0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	{2,6}	1
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	{1,6}	–2
0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	{2,5}	1
0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	{1,3}	–2
0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	{2,4}	1
0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	{1,4}	–2
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	{2,3}	1

The kernel of A, as a linear transformation, is generated by

-2	1	1	1	1	1	1	-2	1	-2	1	-2	1	-2	1
-2	1	1	1	1	1	1	1	-2	1	-2	1	-2	1	-2
1	-2	1	-2	1	-2	1	1	1	1	1	-2	1	1	-2
1	-2	1	1	-2	1	-2	1	1	1	1	1	-2	-2	1
1	1	-2	-2	1	1	-2	-2	1	1	-2	1	1	1	1
1	1	-2	1	-2	-2	1	1	-2	-2	1	1	1	1	1

0 0 0 0 0 0 1 0 0 0 0 0 0 0 0

Orthogonal projection of basis vectors onto the kernel

{4,5}

1 -2 1 1 -2 1 -2 1 1 1 1 1 -2 -2 1

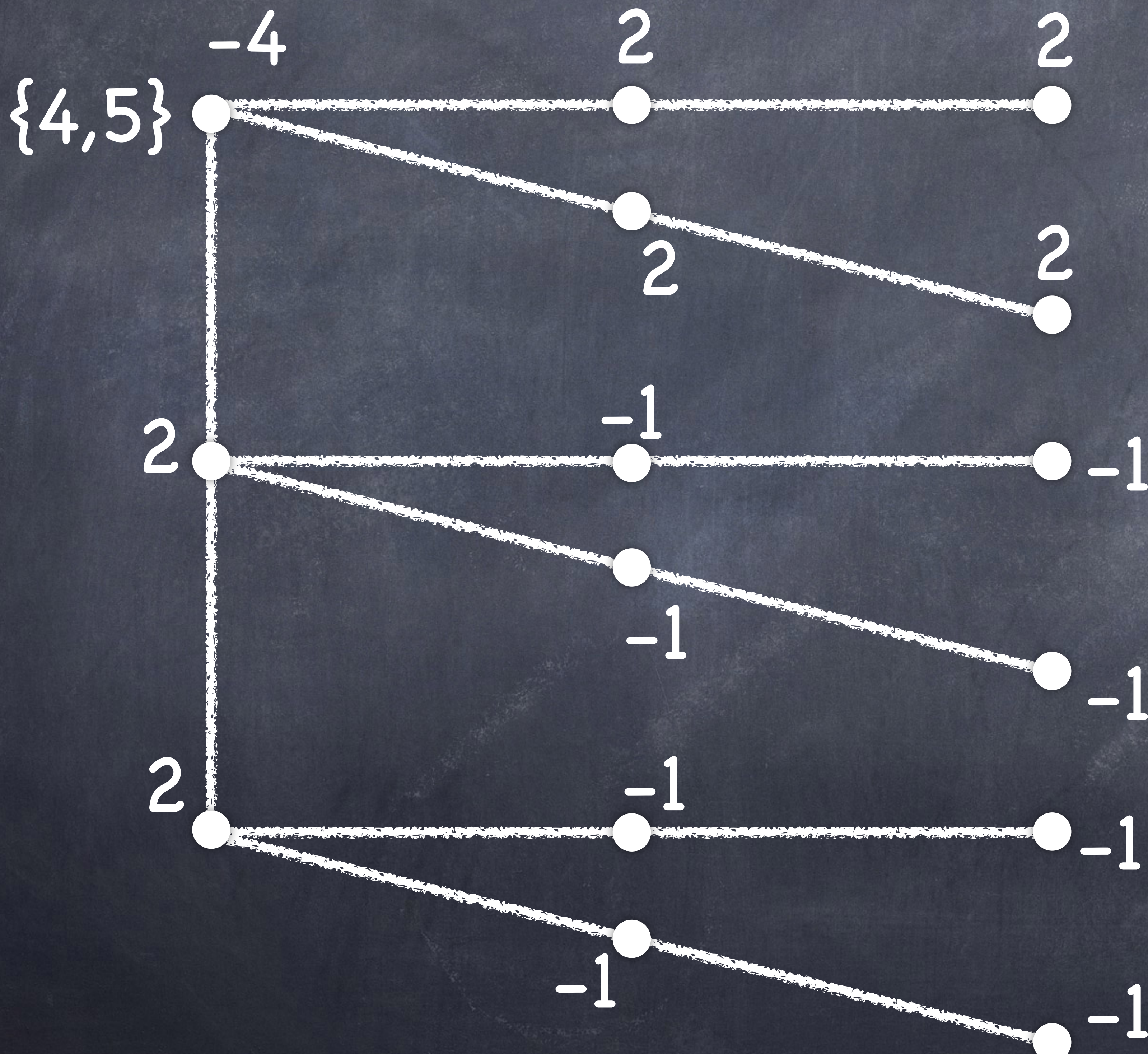
1 1 -2 -2 1 1 -2 -2 1 1 -2 1 1 1 1

2 -1 -1 -1 -1 2 -4 -1 2 2 -1 2 -1 -1 2

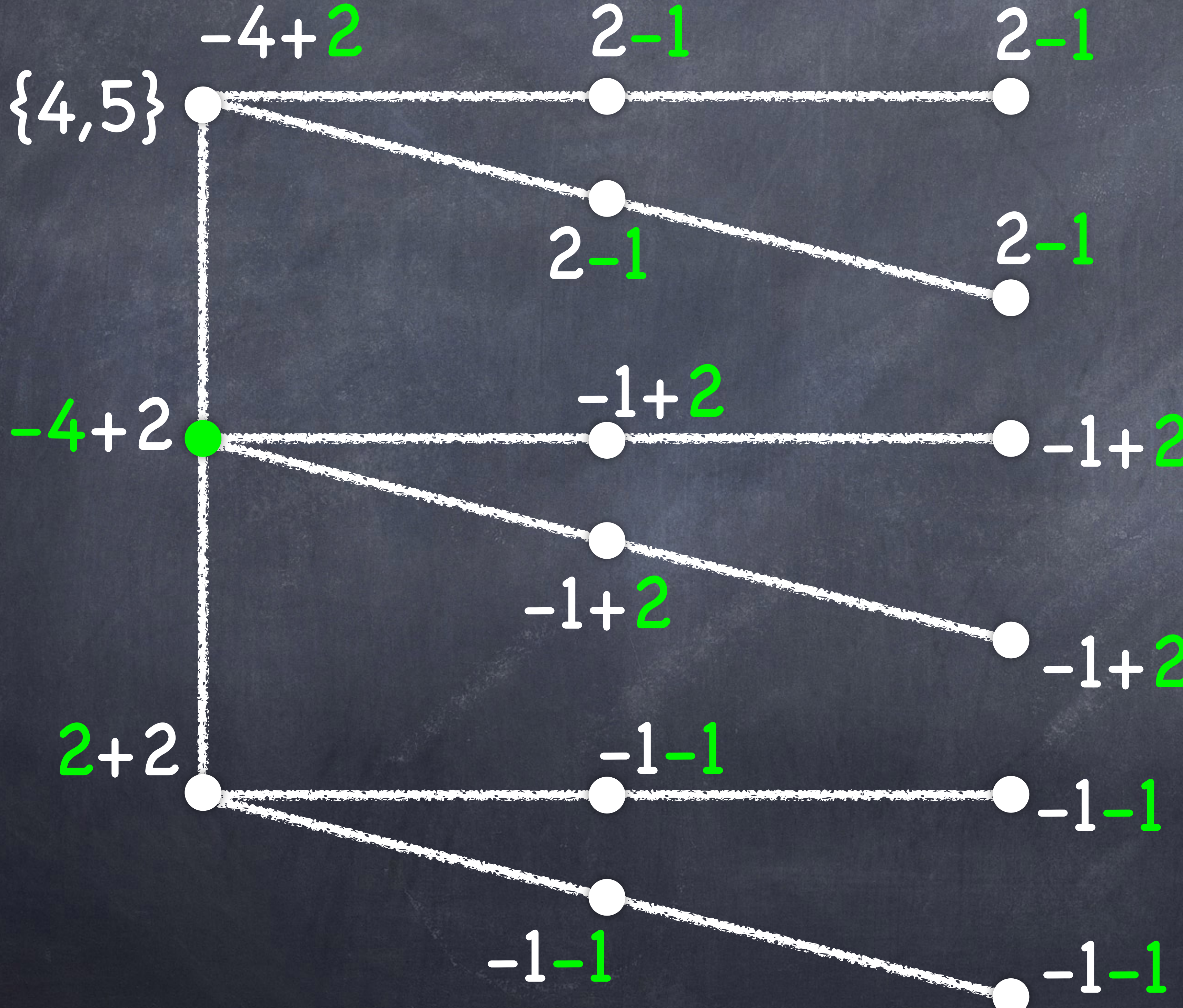
0 0 0 0 0 0 1 0 0 0 0 0 0 0 0

Orthogonal projection of basis vectors onto the kernel

Draw this in the quadrangle:



Draw this in the quadrangle:



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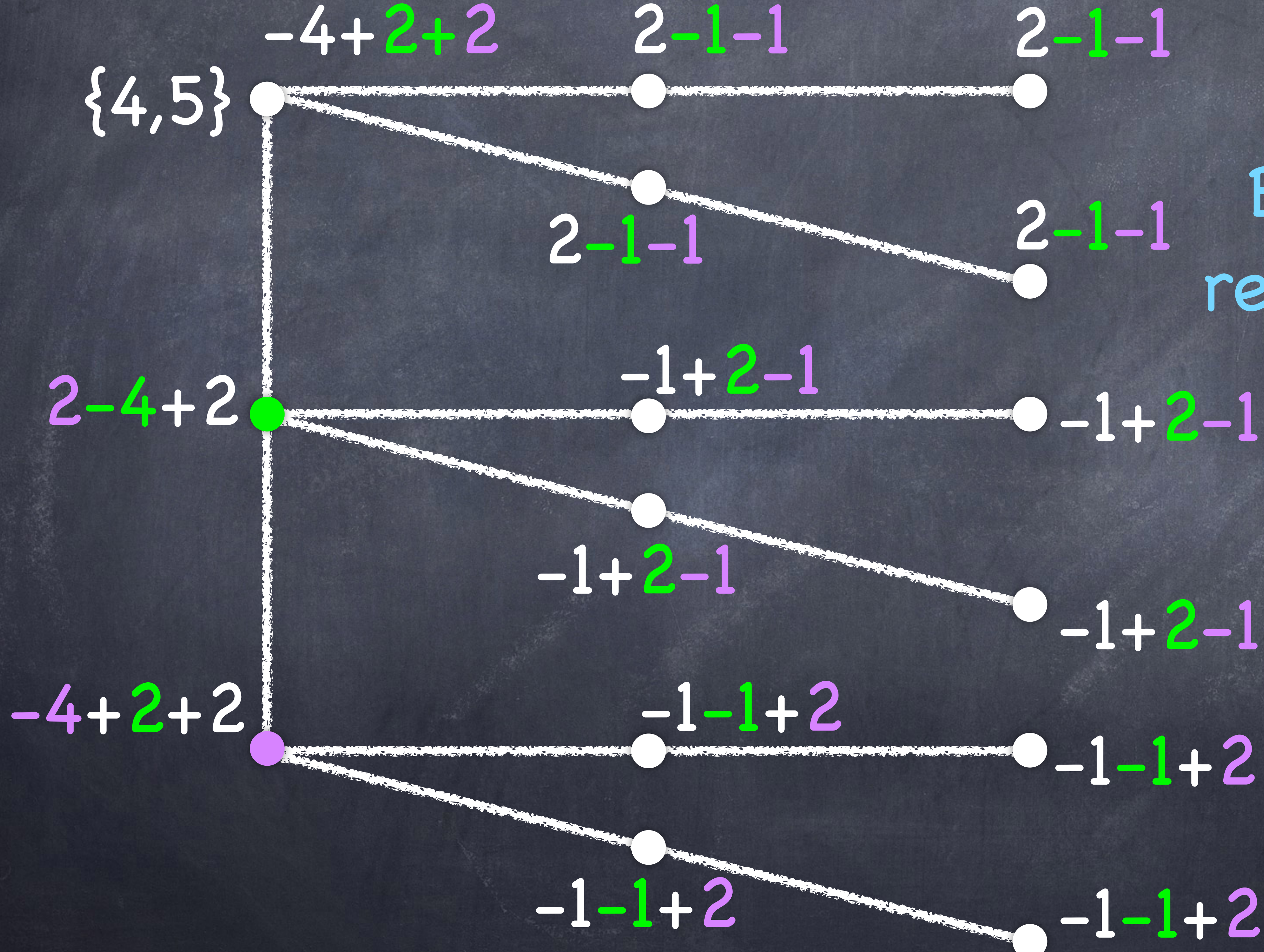


Barycentric representation

This is a general phenomenon:

THEOREM

* Ω is a slim configuration (slim=3points/line)
* Ω is self polar, flag-transitive and primitive
* A is a symmetric incidence matrix of Ω

Then orthogonal projection of basis vectors onto Ker A yields the universal barycentric embedding

Projectively: the projection of the base points from Im A onto Ker A is the universal barycentric embedding.

Question: What if A is nonsingular?

E.g. Desargues configuration

Answer 1: There is no barycentric embedding

Still there are “nice” embeddings

Is there an Answer 2?

Yes!

THEOREM

* Ω is a slim configuration
* Ω is self polar, flag-transitive and imprimitive
* Ω is a double cover of a slim configuration Ω'
* An incidence matrix of Ω' is nonsingular
* A is a singular symmetric incidence matrix of Ω

Then orthogonal projection of basis vectors onto Ker A yields a semi-barycentric embedding of Ω'

Semi-barycentric: $a \pm b \pm c = 0$ for 3 points on a line

Example

Desargues configuration is covered by the dodecahedron geometry:

Point set = {ordered pairs from fixed 5-set {1,2,3,4,5} }

Line set = {domino cycles of length 3}
(eg. {(1,2),(2,3),(3,1)})
= {vertex neighbours in dodecahedron}

Note: |Aut(dodecahedron geometry)|= 2|Aut(dodecahedron graph)|

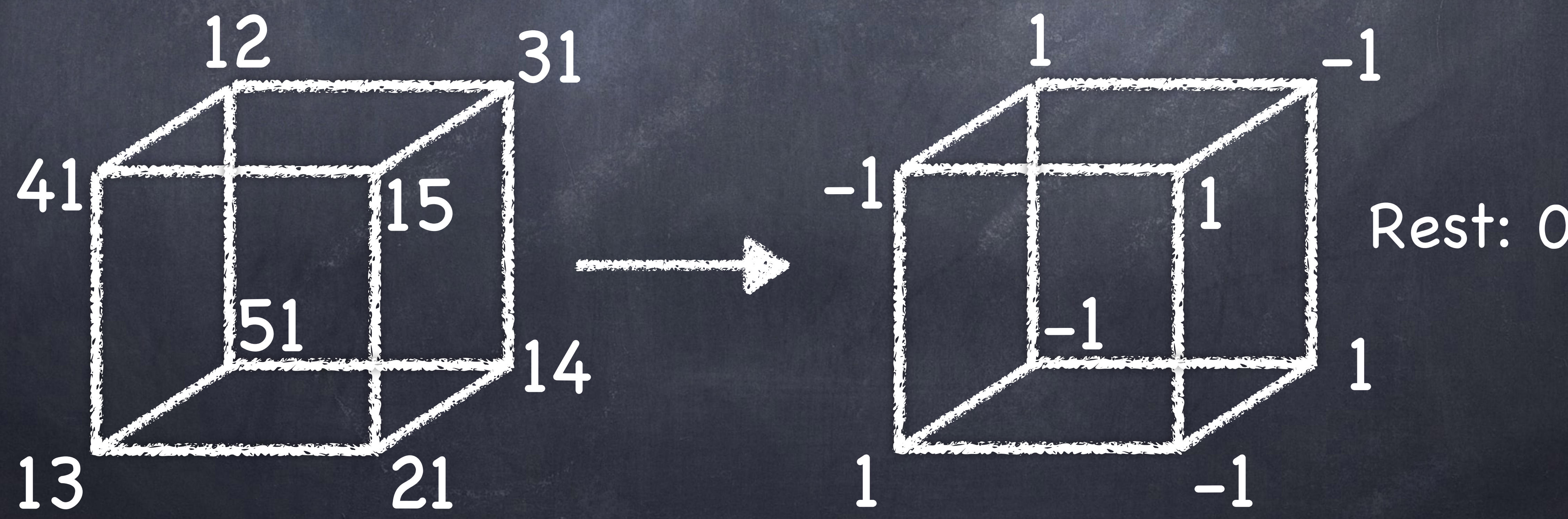
A = symmetric incidence matrix

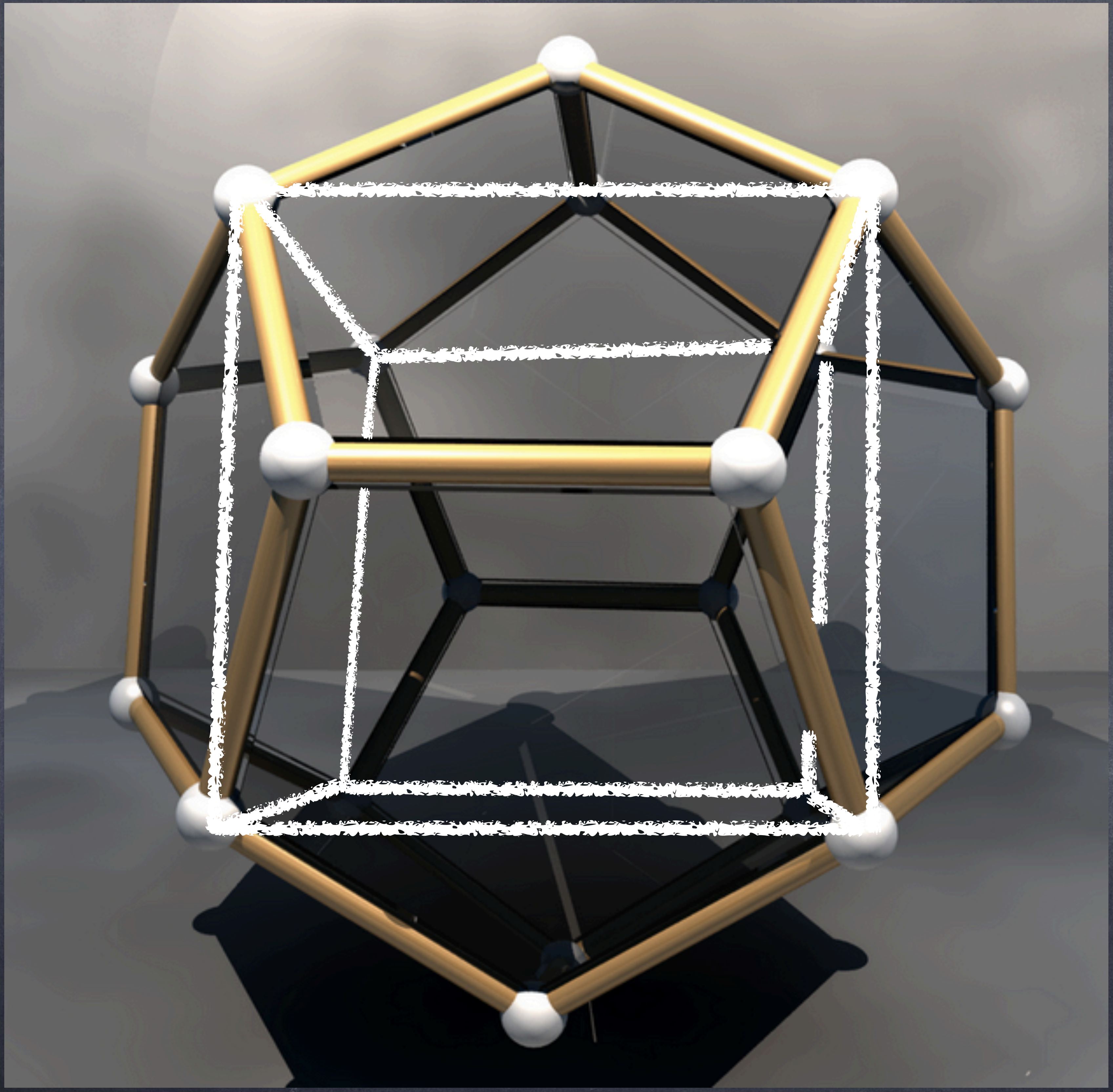
= adjacency matrix of dodecahedron graph

0 is eigenvalue with multiplicity 4

⟶ Ker A has dimension 4

Ker A is determined by inscribed cubes:

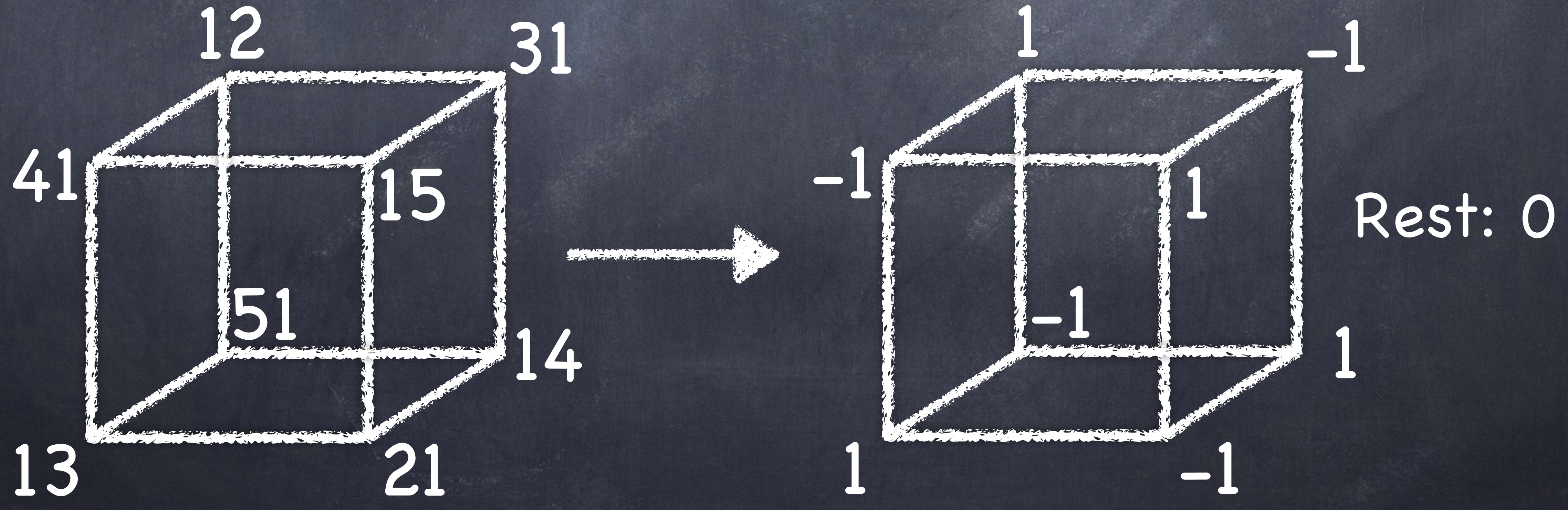




The projection of a basis vector e_{ij} is a multiple of the sum of the two cubes through the corresponding vertex (i,j)

eg. Projection of $6e_{12}$ is

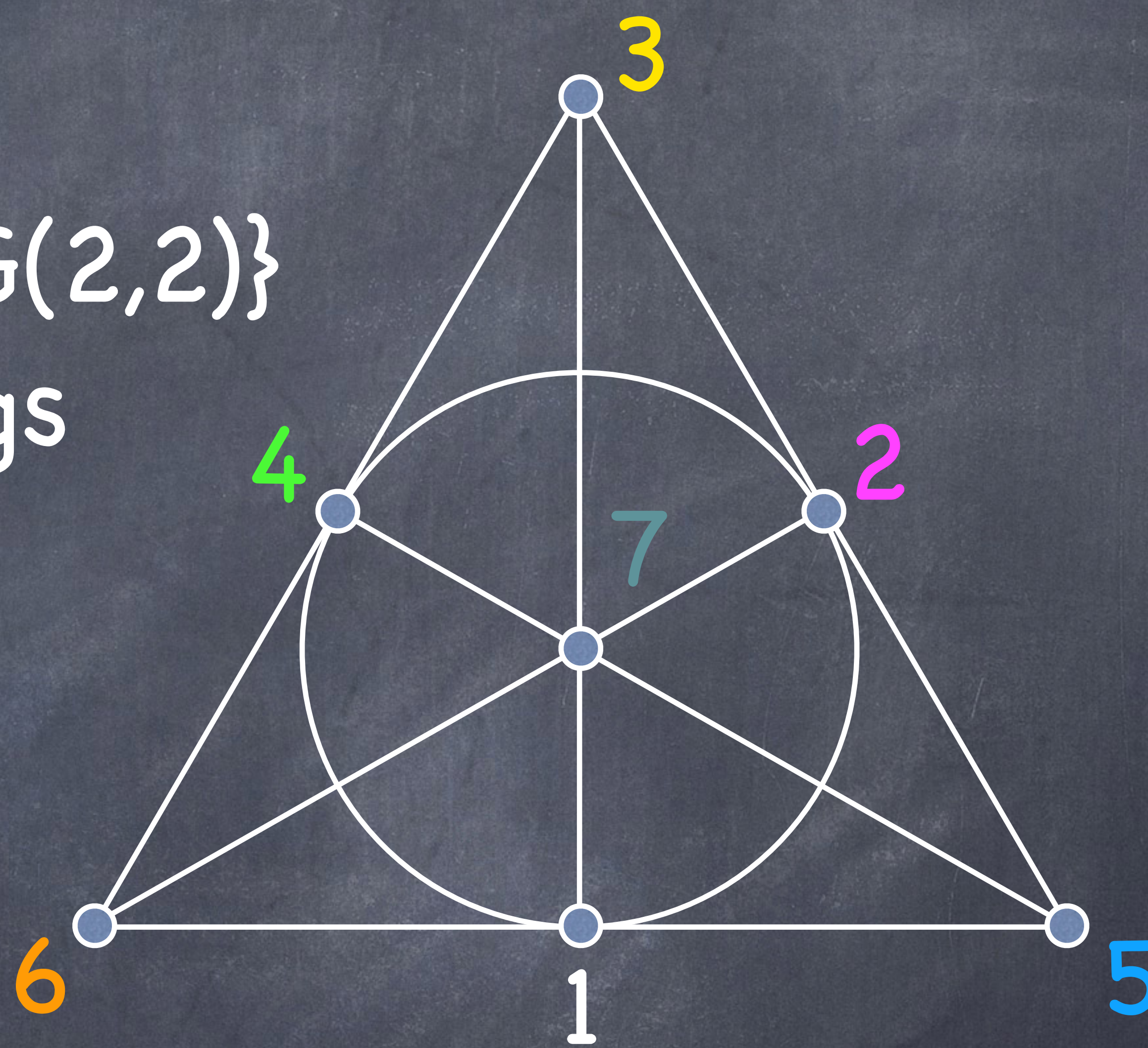
$$\Sigma_{i\neq 1}(e_{1i}-e_{i1}) + \Sigma_{i\neq 2}(e_{i2}-e_{2i})$$



Other example: the projective triangle geometry.

Point set = {antiflags of PG(2,2)}

Line set = {Disjoint antiflags whose union is a triangle}



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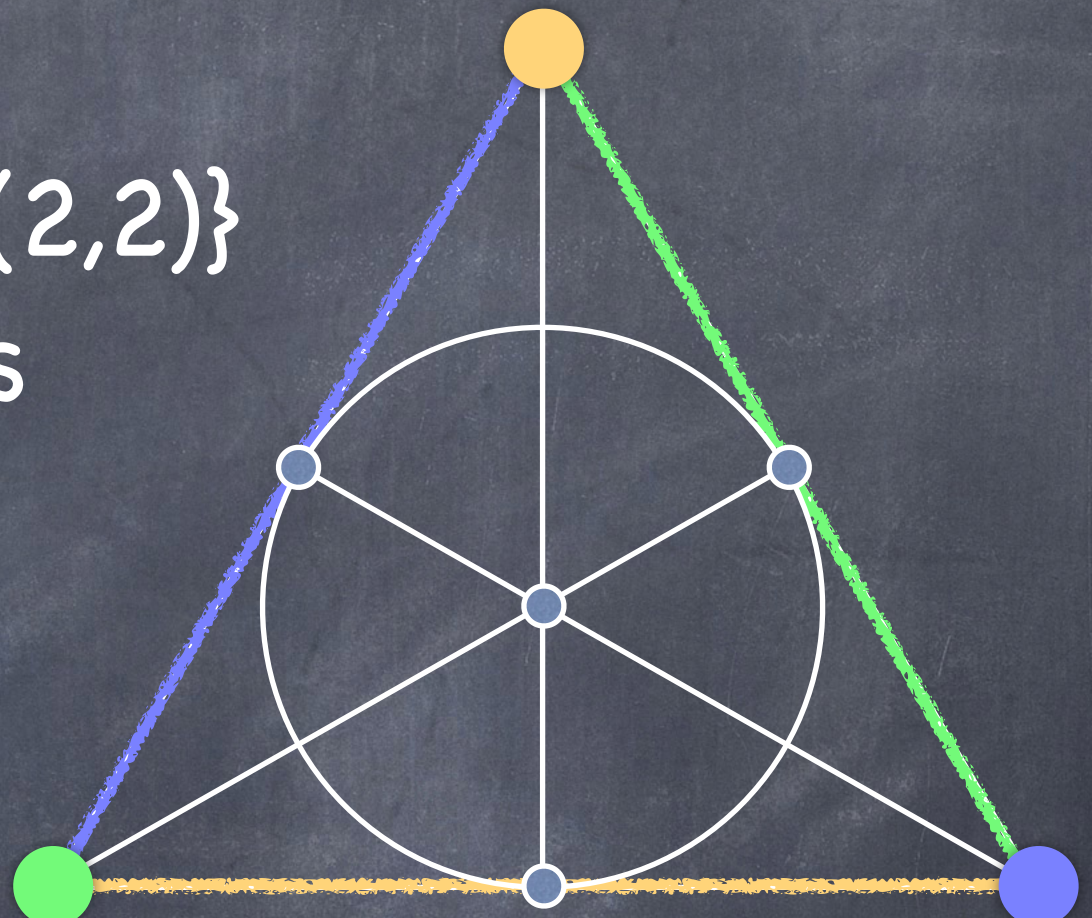

Point set = {antiflags of PG(2,2)}

Line set = {Disjoint antiflags whose union is a triangle}

eg.: (6,{2,3,5})
(5,{3,4,6})
(3,{5,6,1})

This is the neighbourhood geometry of the Coxeter graph

Other example:

The projective triangle geometry.

= neighbourhood geometry of the Coxeter graph

Incidence matrix = adjacency matrix A

A is nonsingular

But ∃ double cover and we obtain semi-barycentric representation in 7-space.

All previous examples have the property that

maximal dimension for a real representation

=

maximal dimension for a GF(2)-representation

Also true for the two slim generalized hexagons H(2) and its dual

Last example: The Biggs-Smith geometry

* Neighbourhood geometry of Biggs-Smith graph on 102 vertices, admits PSL(2,17)
* Incidence matrix=adjacency matrix A of Biggs-Smith graph
* Ker A is 17-dimensional
* Real barycentric embedding in 16-space
* Universal embedding over GF(2) in 18-space

Construction: The Biggs-Smith geometry

* In PG(1,17) given an ordered triple of points (a,b,c) there exist a unique point d and a unique pair {e,f} such that all pairs of {(a,b),(c,d),(e,f)} are harmonic. There are 204 such harmonic triplets with two orbits under PSL(2,17).
* Point set = one orbit of harmonic triplets
* Line set = {partition of PG(1,17) into harmonic triplets}

Construction: Barycentric representation of Biggs-Smith geometry

* Identify each point a of PG(1,17) with a basis vector e_a of $\mathbb{R}^{18}$.
* Let the harmonic triplet {(u,v),(w,x),(y,z)} correspond to the vector $-3(e_u+e_v+e_w+e_x+e_y+e_z)+\sum_k e_k$
* Barycentric representation in hyperplane of $PG(\mathbb{R}^{18})=PG(17,\mathbb{R})$ with equation $\sum_k x_k=0$

Some problems:

* What happens if we do not have flag-transitive group?
* What if the configuration is not self-polar?
* What happens with configurations with more than 3 points per line?

The end

Thank you!