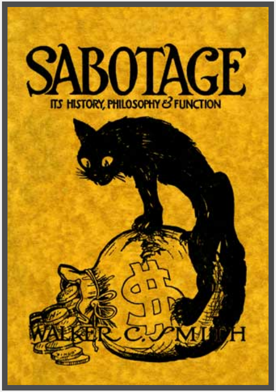
An Evans-style result for block designs



Daniel Horsley (Monash University, Australia)

joint work with

Ajani De Vas Gunasekara

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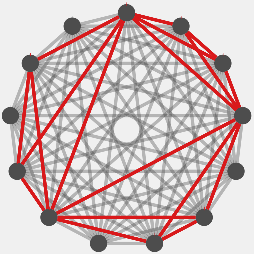

A partial $(13, 4, 1)$-design with three blocks

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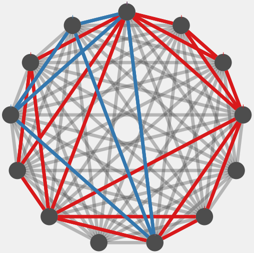

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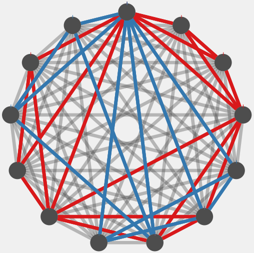

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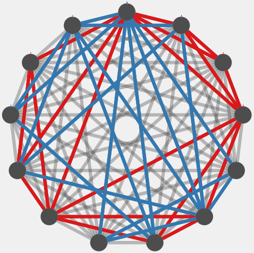

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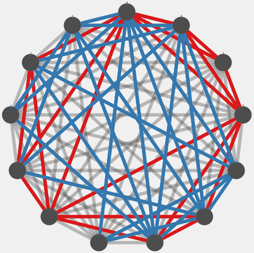

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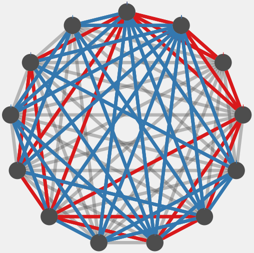

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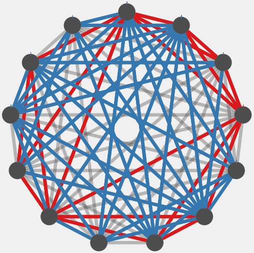

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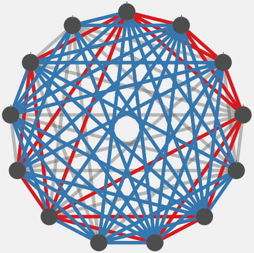

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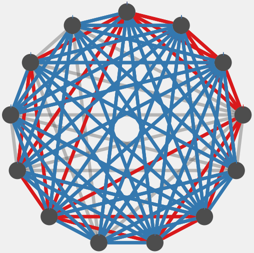

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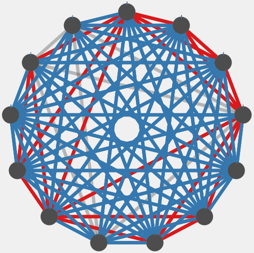

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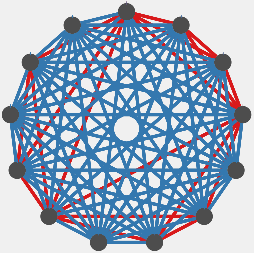

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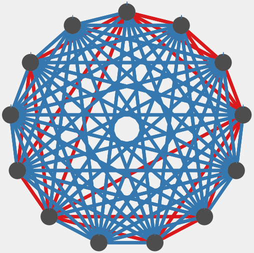

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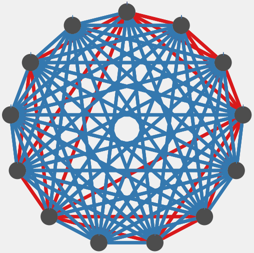

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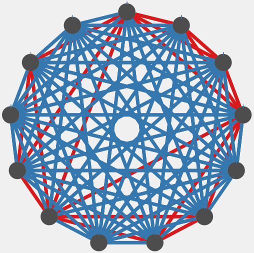

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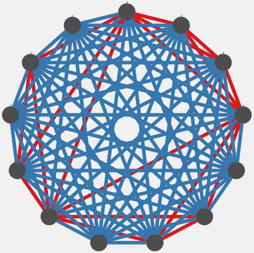

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Let $k \geqslant 3$ be fixed. For all sufficiently large k-admissible n, an $(n,k,1)$-design exists.

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Lots of results on *embedding* partial $(n, k, 1)$-designs.

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Theorem (De Vas Gunasekara, H)

Let $k \geqslant 3$ be fixed. For all k-admissible $n \geqslant k^2 - k + 1$ there is a partial $(n, k, 1)$-design with $\frac{n-1}{k-1} - k + 2$ blocks that is not completable.

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For general k, removing the assumption that n is large would involve solving the existence problem for $(n, k, 1)$-designs.

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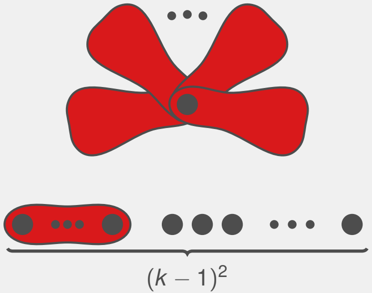
An uncompletable partial $(n, k, 1)$-design with $\frac{n-1}{k-1} - k + 2$ blocks:



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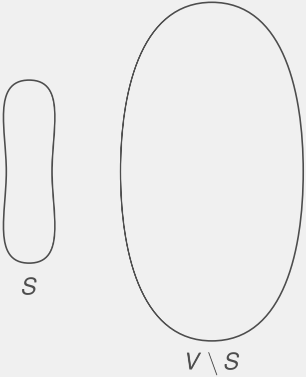
- Let $S = \{x \in V(L) : \deg_L(x) \leqslant (1 - \frac{\gamma}{2})n\}$ and note $|S| \leqslant \frac{1}{4k}\gamma n$.



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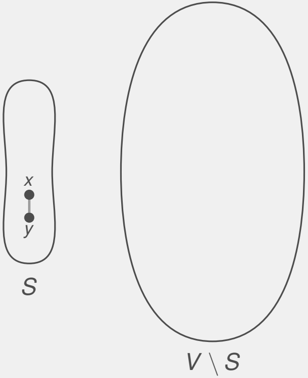

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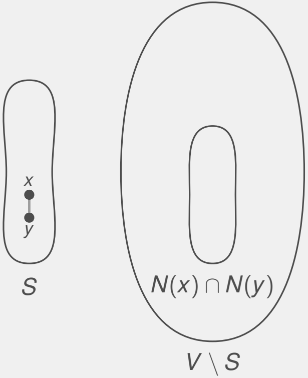

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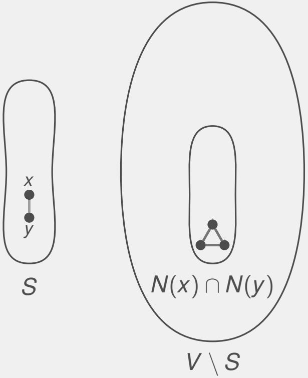

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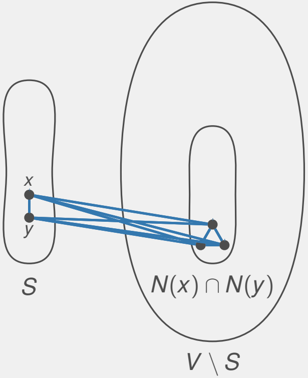

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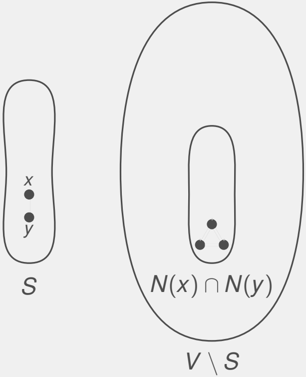

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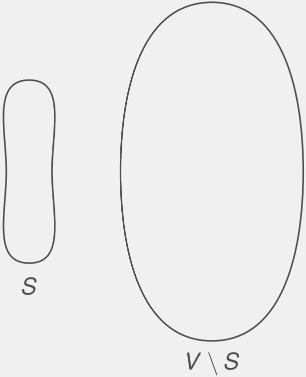

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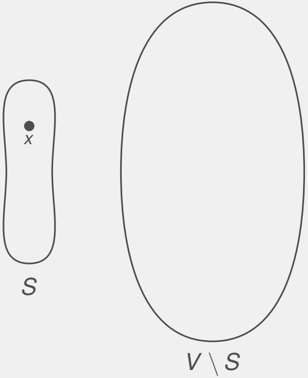

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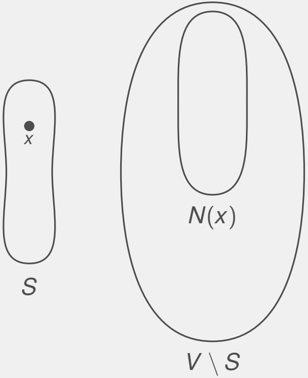

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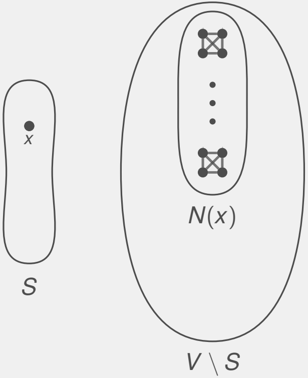

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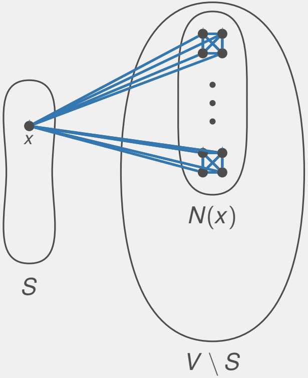

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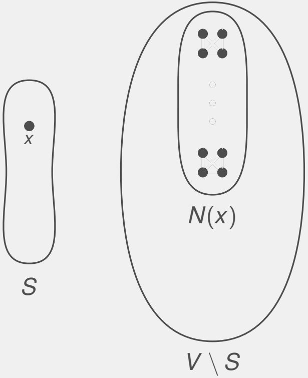

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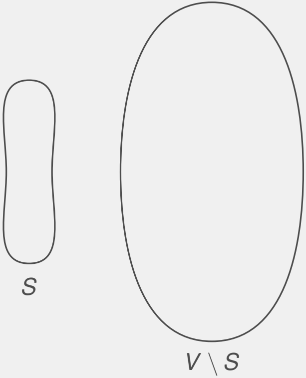

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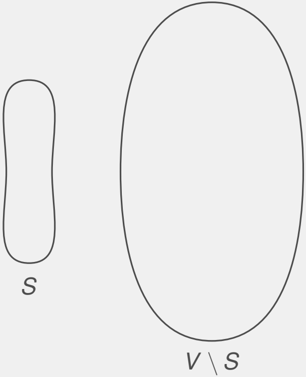

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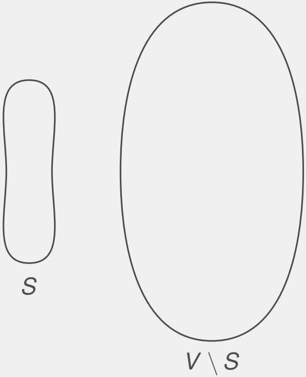
- This is a refinement of an idea used by Nenadov-Sudakov-Wagner.



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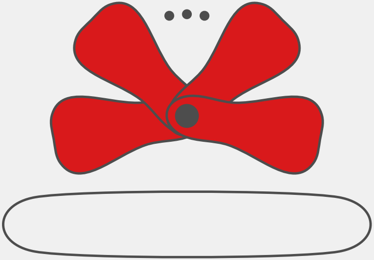

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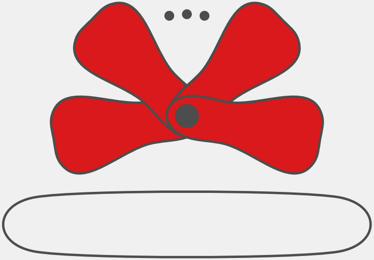

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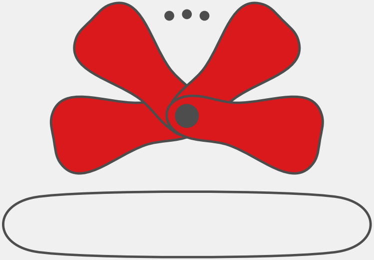

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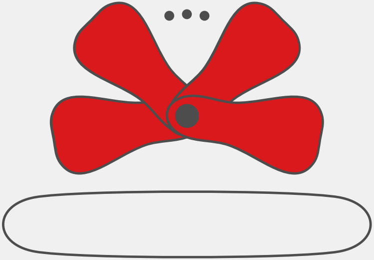

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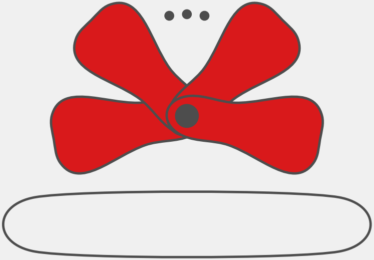

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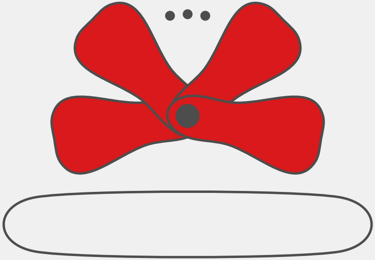

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- Thus each edge is in many triangles and we can use the lemma from the last slide.

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Let $k \geqslant 3$ be fixed. For all sufficiently large k-admissible n, the leave L of any partial $(n, k, 1)$-design has a K_k-decomposition if $|E(L)| > \binom{n}{2} - (\frac{n-1}{k-1} - k + 2)\binom{k}{2}$.

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Each weakening of the assumptions on L requires increasing the lower bound on $|E(L)|$.

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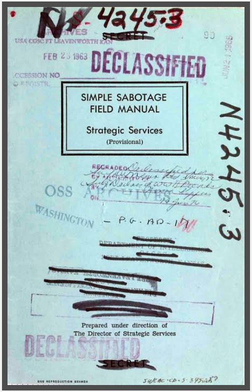
- Very recently, Gruslys and Letzter showed that any graph of order $n \geqslant 7$ with more than $\binom{n}{2} - n + 3$ edges has a **fractional** K_3-decomposition and that this bound is tight. Can similar result be obtained for fractional K_4-decompositions etc?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That’s all.