Special geometries with torus symmetry

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Special geometries

Large symmetry group

Multi-moment maps

A good hierarchy

Nearly Kähler

Unimodular symmetry

SPECIAL GEOMETRIES

RICCI-FLAT GEOMETRIES with closed forms

1. hyperKähler hK_{4k}: $\dim M = 4k$, three symplectic forms $\omega_I, \omega_J, \omega_K$;
2. Calabi-Yau CY_{2m}: $\dim M = 2m$, symplectic ω, complex volume $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$;
3. holonomy G_2: $\dim M = 7$, forms $\varphi \in \Omega^3(M)$, $*\varphi \in \Omega^4(M)$;
4. holonomy $\mathrm{Spin}(7)$: $\dim M = 8$, form $\Omega \in \Omega^4(M)$.

POSITIVE EINSTEIN GEOMETRY nearly Kähler six-manifolds nK_6: $d\omega = 3\psi_+$, $d\psi_- = -2\omega^2$.

Use symmetry to study examples.

LARGE SYMMETRY GROUP

RICCI-FLAT CASES: there are complete cohomogeneity-one examples for each geometry. Early examples

1. hK_{4k}: Calabi (1978) metric on $T^*\mathbb{C}P(2k)$
2. G_2: Bryant and Salamon (1989) metrics on $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{C}P(2))$, $S^3 \times \mathbb{R}^4$
3. $\mathrm{Spin}(7)$: Bryant and Salamon (1989) metric on spin-bundle of S^4.

NEARLY KÄHLER nK_6: homogeneous examples are exactly the six-dimensional three-symmetric spaces (Butruille 2005) constructed by Gray (1972).

G_2, $\mathrm{Spin}(7)$: further cohomogeneity one examples Brandhuber et al. (2001), Cvetič et al. (2002a,b), Bazaĭkin (2007), Bogoyavlenskaya (2013). Foscolo, Haskins, and Nordström (2018) classify all holonomy G_2 with $\mathrm{SU}(2)^2 \times S^1$ symmetry.

nK_6: Foscolo and Haskins (2017) cohomogeneity one examples on S^6 and $S^3 \times S^3$.

MULTI-MOMENT MAPS

G acting (effectively) on M preserving a closed $(p+1)$-form α.
A *multi-moment map* is an equivariant map

$$\nu\colon M \longrightarrow \mathrm{LieKer}(p,\mathfrak{g})^*, \qquad (d\nu)_x(w) = \alpha_x(w,\,\cdot\,),$$

where

$$\mathrm{LieKer}(p,\mathfrak{g}) := \ker[\,\cdot\,,\,\cdot\,]\colon \Lambda^p\,\mathfrak{g} \longrightarrow \Lambda^{p-1}\,\mathfrak{g}$$

$$w = \sum_{i=1}^{k} X_1^k \wedge \cdots \wedge X_p^k \in \mathrm{LieKer}(p,\mathfrak{g}).$$

For G Abelian, $\mathfrak{g} = \mathbb{R}^r$, $\mathrm{LieKer}(p,\mathfrak{g}) = \Lambda^p\,\mathfrak{g}^* = \mathbb{R}^N$, $N = \binom{r}{p}$.

(Madsen and Swann 2012)

A GOOD HIERARCHY

Dimension matching: $\dim(M/T^r) = \dim \operatorname{LieKer}(p, \mathfrak{g})$.

$$\begin{gathered} \mathrm{hK}_1 \subset \mathrm{CY}_3 \subset G_2 \subset \mathrm{Spin}(7) \\ T^1 \leqslant T^2 \leqslant T^3 \leqslant T^4 \end{gathered}$$

MASTER CASE $\mathrm{Spin}(7)$ $\nu\colon M^8 \to \mathbb{R}^4$.
Structure on free part: given by positive definite $V = (g(X_i, X_j))^{-1}$ satisfying

$$\sum_{i,j=1}^{4} \frac{\partial^2}{\partial \nu_i \partial \nu_j}(V_{ij}V_{ab} - V_{ia}V_{jb}) = 0, \qquad \sum_{i=1}^{4} \frac{\partial V_{ia}}{\partial \nu_i} = 0.$$

Local solutions exist in abundance.
Singular orbit have stabiliser a subtorus of T^4: only T^1 or T^2.
Image under ν of singular orbits is an affine trivalent graph in $\mathbb{R}^4$ with rational slopes and zero-tension condition. (Madsen and Swann 2019a,b)
EXISTENCE OF COMPLETE METRICS? hK_1, CY_3, G_2 yes. $\mathrm{Spin}(7)$ unknown.

NEARLY KÄHLER nK_6

T^2 SYMMETRY $\nu = \omega(X_1, X_2)\colon M \to \mathbb{R}$. (Russo and Swann 2019)
For M compact, $\nu(M) = [a, b] \subset \mathbb{R}$, $a < 0 < b$.
Stabilisers of dim > 0 only occur in $\nu^{-1}(0)$.

For regular value t, $N = \nu^{-1}(t)/T^2$ is a three-manifold with global coframe $\alpha_0, \alpha_1, \alpha_2$ satisfying

$$d\alpha_0 = f\alpha_1 \wedge \alpha_2, \quad d(f\alpha_1) \wedge \alpha_0 = 0 = d(f\alpha_2) \wedge \alpha_0.$$

The nearly Kähler geometry is recovered by geometric flow.

N is homogeneous with left-invariant data if and only if N is three-dimensional unimodular group that is not Abelian.

T^3 SYMMETRY $\nu\colon M \to \mathbb{R}^3$. Moroianu and Nagy (2019).
Occurs for three-symmetric structure on $S^3 \times S^3$.
Are there other examples?

UNIMODULAR SYMMETRY OF G_2-STRUCTURES

Approach of Chihara (2019).
M^7 torsion-free G_2. Three-form φ, four-form $*\varphi$.

Free Lagrangian action of three-dimensional G: $\varphi(X_1, X_2, X_3) = 0$.
Multi-moment map $\nu\colon M \to \mathbb{R}$, $d\nu = *\varphi(X_1, X_2, X_3, \cdot\,)$: closed if and only if G is unimodular.

Have a G-bundle $\nu^{-1}(t) \to N^3 = \nu^{-1}(t)/G$ with Riemannian connection $a = a^i X_i$ and solder form $b = b^i X_i = \varphi(X_j, X_k, \cdot\,)X_i$. Get a geometric flow for $V = (g(X_i, X_j))^{-1}$.

$$\varphi = (\operatorname{adj} V)_{ij} a^i b^j d\nu - (\det V) b^{123} + b^i a^{jk}.$$

Chihara (2019): T^3 and $SO(3)$. Boye (2021): general G.

$G = SU(2)$: direct solutions on $N^3 = SU(2)$ with $b^i = f\sigma^i$, $V = (f_\nu/f)^{1/2}\mathrm{Id}$, including Bryant-Salamon cone metric.
Cf. Karigiannis and Lotay (2021) and Kawai (2018).

REFERENCES I

Bazaĭkin, Y. V. (2007), 'On new examples of complete noncompact metrics with Spin(7) holonomy group', *Sibirsk. Mat. Zh.* 48 (1): 11–32.

Bogoyavlenskaya, O. A. (2013), 'On a new family of complete Riemannian metrics on $S^3 \times \mathbb{R}^4$ with holonomy group G_2', *Sibirsk. Mat. Zh.* 54 (3): 551–62.

Boye, E. T. (2021), 'Exceptional holonomy and group actions', Master's thesis (Aarhus University), 92 pp.

Brandhuber, A. et al. (2001), 'Gauge Theory at Large N and New G_2 Holonomy Metrics', *Nuclear Phys. B* 611 (1-3): 179–204.

Bryant, R. L. and Salamon, S. M. (1989), 'On the construction of some complete metrics with exceptional holonomy', *Duke Math. J.* 58: 829–50.

Butruille, J.-B. (2005), 'Classification des variétés approximativement kähleriennes homogènes', *Ann. Global Anal. Geom.* 27 (3): 201–25.

Calabi, E. (1978), 'Métriques kähleriennes et fibrés holomorphes.', *Ann. Scient. Éc. Norm. Sup.* 12: 269–94.

References II

Chihara, R. (2019), 'G_2-metrics arising from non-integrable special Lagrangian fibrations', *Complex Manifolds*, 6 (1): 348–65.

Cvetič, M. et al. (2002a), 'Cohomogeneity one manifolds of $\mathrm{Spin}(7)$ and G_2 holonomy', *Ann. Physics*, 300 (2): 139–84.

Cvetič, M. et al. (2002b), 'New complete noncompact $\mathrm{Spin}(7)$ manifolds', *Nuclear Phys. B* 620 (1-2): 29–54.

Foscolo, L. and Haskins, M. (2017), 'New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$', *Ann. of Math. (2)*, 185 (1): 59–130.

Foscolo, L., Haskins, M., and Nordström, J. (2018), *Infinitely many new families of complete cohomogeneity one G_2-manifolds: G_2 analogues of the Taub-NUT and Eguchi-Hanson spaces*, May, arXiv: `1805.02612 [math.DG]`.

Gray, A. (1972), 'Riemannian manifolds with geodesic symmetries of order 3', *J. Differential Geometry*, 7: 343–69.

Karigiannis, S. and Lotay, J. D. (2021), 'Bryant-Salamon G_2 manifolds and coassociative fibrations', *J. Geom. Phys.* 162: 104074, 60.

References III

Kawai, K. (2018), 'Cohomogeneity one coassociative submanifolds in the bundle of anti-self-dual 2-forms over the 4-sphere', *Comm. Anal. Geom.* 26 (2): 361–409.

Madsen, T. B. and Swann, A. F. (2012), 'Multi-moment maps', *Adv. Math.* 229: 2287–309.

Madsen, T. B. and Swann, A. F. (2019a), 'Toric geometry of G_2-manifolds', *Geom. Topol.* 23 (7): 3459–500.

Madsen, T. B. and Swann, A. F. (2019b), 'Toric geometry of $\mathrm{Spin}(7)$-manifolds', to be published in *Int. Math. Res. Not. IMRN*.

Moroianu, A. and Nagy, P.-A. (2019), 'Toric nearly Kähler manifolds', *Ann. Global Anal. Geom.* 55 (4): 703–17.

Russo, G. and Swann, A. F. (2019), 'Nearly Kähler six-manifolds with two-torus symmetry', *J. Geom. Phys.* 138: 144–53.