Numerical scheme for an equation on a graph for a flow in a tube structure

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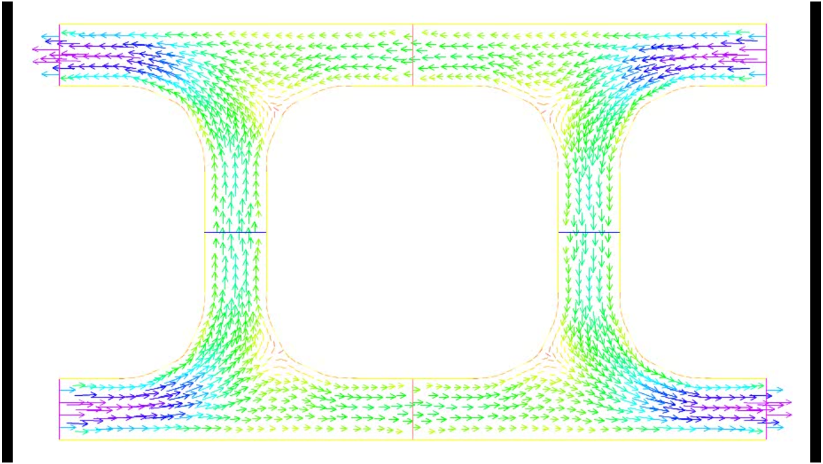
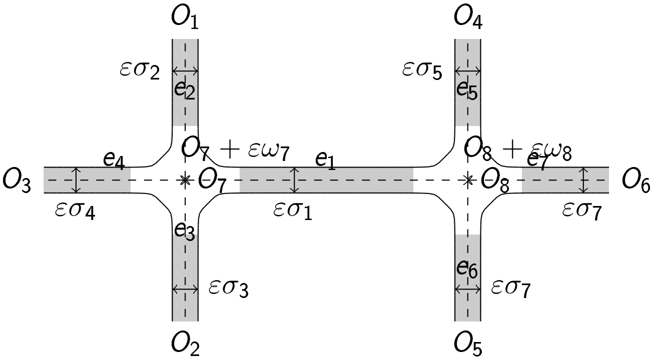
Fluid in a network of thin tubes





- $O_1, O_2, \ldots, O_N$ vertices, $\omega_1, \ldots, \omega_N \subset \mathbb{R}^3$ bounded open sets such that $(0,0,0) \in \omega_n$.
- For $m \in \{1, \ldots, M\}$, define the edge $e_m = [O_{i_m}, O_{k_m}]$ of length l_m.
- $\mathcal{R}_m$ displacements such that $\mathcal{R}_m(0,0,0) = O_{i_m}, \mathcal{R}_m(l_m, 0, 0) = O_{k_m}$.
- $\sigma_1, \ldots, \sigma_M \subset \mathbb{R}^2$ bounded open sets.

Assume that $\Omega = \cup_{m=1}^{M} \mathcal{R}_m(]0, l_m[\times \varepsilon\sigma_m) \cup \cup_{n=1}^{N}(O_n + \varepsilon\omega_n)$ is smooth for small ε except at the ends of unconnected tubes.

Navier-Stokes equations

$$\begin{cases} u = 0 & \text{on walls of tubes and junctions} \\ u = g^{(e_i)} & \text{on unconnected end of tube } e_i \\ \operatorname{div} u = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = \frac{f - \nabla p}{\rho_0} & \text{on } \Omega \\ u(.,0) = 0 & \text{on } \Omega \end{cases} \quad (1)$$

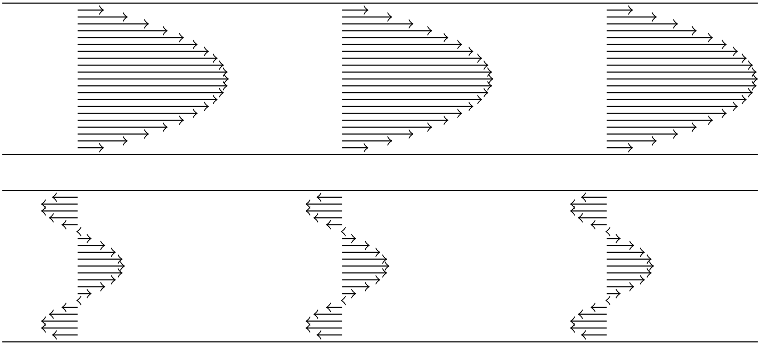
Assumptions:

- The graph scales as 1.
- The width of the pipes scales as ε.
- The time scales as ε
- The norms in $W^{2,\infty}L^2$, $W^{1,\infty}H^1$, $L^\infty H^2$ of the velocity at the extremities of the network scale as $\varepsilon^{\frac{n-1}{2}}$, $\varepsilon^{\frac{n-3}{2}}$, $\varepsilon^{\frac{n-5}{2}}$.

Studied in:

- Panasenko & Pileckas 2014, 2015, 2015

Flow in an infinite thin tube $\Omega = \mathbb{R} \times \sigma$ (Pileckas 2006)

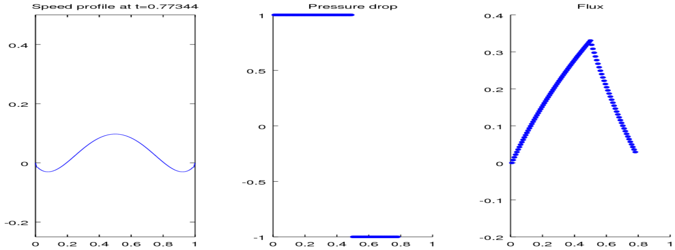


Let us seek solution in the form $u(x, y, z, t) = V(y, z, t)\vec{e}_x$ and $p(x, y, z, t) = q(t)x + r(t)$.

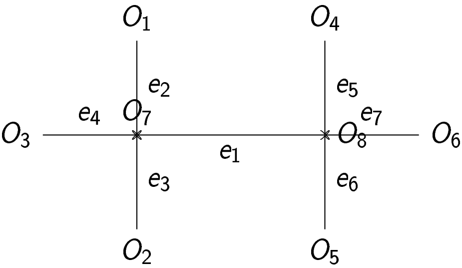
Pressure gradient - flux operator

Let $L^{(\sigma)} : \begin{cases} L^2(0,\infty) \to H_0^1(0,\infty) \\ L^{(\sigma)} q = \Phi = \int_\sigma V = \int_0^t K^{(\sigma)}(t-\tau) q(\tau) \mathrm{d}t \end{cases}$ be the operator connecting the pressure gradient to the flux through the pipe, where:

$$\begin{cases} \frac{\partial U}{\partial t} - \nu \Delta_{y,z} U = 0 & \text{on } \sigma \\ U(.,0) = \frac{1}{\rho_0} & \text{on } \sigma \\ U = 0 & \text{on } \partial\sigma \\ K(t) = \int_\sigma U(.,t) \end{cases}.$$



Flow in a web of thin tubes



- $\mathcal{B} = \cup_{m=1}^{M} e_m = \cup_{m=1}^{M} [O_{i_m} O_{k_m}]$ is a graph with Lebesgue measure.
- $L^2(\mathcal{B}) = \{u | \int_{\mathcal{B}} |u|^2 < +\infty\}$ and $H^1(\mathcal{B}) = \{f | \int_{\mathcal{B}} |f|^2 + |\frac{\partial f}{\partial x}|^2 < +\infty, f(O_1) = 0\}$ where $\frac{\partial}{\partial x}$ is the directionnal derivative along $\frac{\overrightarrow{O_{i_m} O_{k_m}}}{l_m}$ on $]O_{i_m} O_{k_m}[$.
- $\Psi_n : [0, T] \to \mathbb{R}$ be the flux of fluid at O_n coming from outside Ω.
- $f : \mathcal{B} \times [0, T] \to \mathbb{R}$ be a forcing along the pipes.

Flow in a net of tubes

Assumptions:

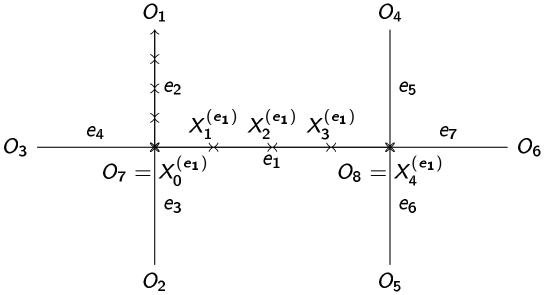
- $\Psi \in H^1_{00}(0,T)^N$ and $f \in H^1_{00}(0,T,L^2(\mathcal{B}))$. (The subscript 00 means that Ψ, f are zero at initial time)
- $\int_{\mathcal{B}} f + \sum_{l=1}^{N} \Psi_l = 0$

The fluid in the net of tubes can be approximated by:

$$\begin{cases} \Psi_n(t) + \sum\limits_{[O_n,O_{\tilde{n}}]=e_m} L^{(\sigma_m)} D_{\frac{\overrightarrow{O_n O_{\tilde{n}}}}{l_m}} p(O_n,t) = 0 \quad \text{(Kirchoff condition)} \\ -\frac{\partial}{\partial x} L^{(\sigma_m)} \frac{\partial}{\partial x} p = f \text{ on } e_m \\ p \text{ continuous on } \mathcal{B} \\ p(O_1,t) = 0 \end{cases}$$

where $D_{\vec{v}}$ denotes the directionnal derivative along $\vec{v}$ and $(L^{(\sigma_m)} q)(t) = \int_0^t K^{(\sigma_m)}(t-\tau) q(\tau) \mathrm{d}t$.
Kirchoff condition expresses that the sum of the fluxes arriving from tube m at a junction n should be zero.

Graph discretization



- Take a subdivision of $[0, T]$ of step k: $t_0 = 0 < t_1 < t_2 < \ldots < t_Q = T,\ t_q = qk,\ Q = \frac{T}{n}$.
- For $e_j \in \{e_1, \ldots, e_M\}$, take a subdivision into $S^{(e_j)}$ segments of step $h^{(e_j)} = \frac{|e_j|}{S^{(e_j)}}$. Let us denote $X_s^{(e_j)} = \frac{S^{(e_j)}-s}{S^{(e_j)}} O_{i_j} + \frac{s}{S^{(e_j)}} O_{k_j}$ the nodes of this subdivision for $s \in \{0, 1 \ldots, S^{(e_j)}\}$.
- Let $h = \max_{1 \leq j \leq M} h^{(e_j)}$.

Finite difference scheme

Let $P^{(e)}_{s,q}$ be an approximation of $P(X^{(e)}_s, t_q + \frac{k}{2})$. Consider:

$$\Psi_\ell(t_{q+1}) = \sum_{\substack{1 \le j \le M, \\ X_s^{(e_j)} = O_\ell \in e_j, \\ |s-\tilde{s}| = 1, 0 \le \tilde{s} \le S^{(e_j)}}} \left[\frac{-h^{(e_j)}}{2} F(X_s^{(e_j)}, t_{q+1}) - k \sum_{\tilde{q}=0}^{q} K^{(\sigma_j)}_{q-\tilde{q}} \frac{P^{(e_j)}_{\tilde{s},\tilde{q}} - P^{(e_j)}_{s,\tilde{q}}}{h^{(e_j)}} \right] \text{ if } \begin{cases} 2 \le \ell \le N \\ 0 \le q < Q \end{cases},$$

$$F(X_s^{(e_j)}, t_{q+1}) = -k \sum_{\tilde{q}=0}^{q} K^{(\sigma_j)}_{q-\tilde{q}} \frac{P^{(e_j)}_{s+1,\tilde{q}} - 2P^{(e_j)}_{s,\tilde{q}} + P^{(e_j)}_{s-1,\tilde{q}}}{(h^{(e_j)})^2} \text{ if } \begin{cases} 1 \le j \le M \\ 0 < s < S^{(e_j)} \\ 0 \le q < Q \end{cases},$$

$P^{(e)}_{s,q} = P^{(\tilde{e})}_{\tilde{s},q}$ if $X^{(e)}_s = X^{(\tilde{e})}_{\tilde{s}}, 0 \le q \le Q,$

$P^{(e)}_{0,q} = 0$ if $X^{(e)}_0 = O_1, 0 \le q \le Q,$

where $K^{(\sigma_j)}_q \simeq \frac{1}{k} \int_{t_q}^{t_{q+1}} K^{(\sigma_j)}(t) \mathrm{d}t$ for $0 \le q \le Q$, $1 \le j \le M$.

Accuracy of the kernel approximation

In order to measure the accuracy of the approximation of the discretized kernel $(K_q^{(\sigma)})_q$, let us introduce:

$$\theta(k) = \max_{1\leq j\leq M} |K_0^{(\sigma_j)} - \frac{1}{k}\int_0^k K^{(\sigma_j)}(t)\mathrm{d}t| + \sum_{q=1}^{Q-1} |K_q^{(\sigma_j)} - K_{q-1}^{(\sigma_j)} - \frac{1}{k}\int_{t_q}^{t_{q+1}} K^{(\sigma_j)}(t) - K^{(\sigma_j)}(t-k)\mathrm{d}t|$$

Notice that it is a kind of discrete $W^{1,1}$ norm.

Weak form

- Let $p, \psi \in L^2(0, T, H^1(\mathcal{B}))$. Let us denote:

$$a(p,\psi) = \int_{[0,T]\times\mathcal{B}} \frac{\partial^2 (L^{(\overline{\sigma})} p)}{\partial x \partial \tau} \frac{\partial \psi}{\partial x}$$

$$b(\psi) = \int_{[0,T]\times\mathcal{B}} \frac{\partial f}{\partial \tau}\psi + \int_0^T \sum_{n=1}^{N} \frac{\partial \Psi_n}{\partial \tau}\psi(O_n, .)$$

Then, the weak form for the continuous asymptotic problem is, find $p \in L^2(0, T, H^1(\mathcal{B}))$ such that:

$$\forall \psi \in L^2(0, T, H^1(\mathcal{B})), \quad a(p,\psi) = b(\psi)$$

Lax-Milgram theorem can be used to prove the existence and unicity of a solution.

Discrete weak form (Galerkin method)

- Let $\mathbb{P}^1_h(\mathcal{B})$ be a the subspace of $H^1(\mathcal{B})$ of continuous functions which are piecewise linear over the subdivision of $\mathcal{B}$.
- Let $V_{h,k} = \mathbb{P}^0_k(0, T, \mathbb{P}^1_h(\mathcal{B}))$ be the set of piecewise constant functions over the subdivision of $[0, T]$ with values in $\mathbb{P}^1_h(\mathcal{B})$.
- Let us take $K^{(\sigma)}_q = 0$ when $q < 0$. For $p, \psi \in V_{h,k}$, let us denote:

$$\tilde{a}(p, \psi) = \int_{\mathcal{B}} k \sum_{q=0}^{Q-1} \sum_{\tilde{q}=0}^{Q-1} (K_{q-\tilde{q}} - K_{q-\tilde{q}-1}) \frac{\partial p}{\partial x}(\cdot, \frac{t_{\tilde{q}} + t_{\tilde{q}+1}}{2}) \frac{\partial \psi}{\partial x}(\cdot, \frac{t_q + t_{q+1}}{2})$$

- Let $p_{h,k} \in \mathbb{V}_{h,k}$ such that $p_{h,k}(X^{(e_j)}_s, \frac{t_q+t_{q+1}}{2}) = P^{(e_j)}_{s,q}$.

Then $p_{h,k}$ is a solution to:

$$\forall \psi \in V_{h,k}, \quad \tilde{a}(p_{h,k}, \psi) = \tilde{b}(\psi)$$

where $\tilde{b}$ is a good approximation of b. Besides $\|\tilde{a} - a\| \leq \theta(k)$.

Stability condition

A sufficient condition for existence and uniqueness for the discrete solution is the continuity and coercivity of the discrete form, uniformly when $(h, k) \to (0, 0)$.
Let α_T be coercivity constant of a.
Here are two sufficient criterions:

- If $\theta(k) < \mu < \alpha_T$, then $\tilde{a}$ is $\alpha_T - \mu$-coercive.
- If there exists $C, E, T_m \in \mathbb{R}^{+*}$ and $(K_q^{(\sigma)})_{q\in\mathbb{Z}}$ such that:
  - $0 \leq K_q^{(\sigma)} \leq C$ if $q \geq 0$,
  - $K_{q+1}^{(\sigma)} - 2K_q^{(\sigma)} + K_{q-1}^{(\sigma)} \geq 0$ if $q \geq 1$,
  - $E \leq \frac{K_{q+1}^{(\sigma)} - 2K_q^{(\sigma)} + K_{q-1}^{(\sigma)}}{k^2}$ if $T_m \leq qk \leq 2T_m$.

  then, for $k < \min\{T_m, T\}$, $\tilde{a}$ is $\frac{\tilde{C}}{T^2}$-coercive, with $\tilde{C}$ independent of T, h, k.

Error estimate

According to Céa's lemma:

$$\|p_{h,k} - P\|_{L^2(0,T,H^1(\mathcal{B}))} \leq \frac{C}{\alpha_T}\Bigg[\inf_{\psi \in V_{h,k}} \|P - \psi\|_{L^2(0,T,H^1(\mathcal{B}))} + \frac{\theta(k)}{\alpha_T - \theta(k)}\|P\|_{L^2(0,T,H^1(\mathcal{B}))} + \|b - \tilde{b}\|\Bigg]$$

If $\Psi_l \in H^2_{00}(0,T)$ and $f \in H^2_{00}(0,T,H^2_{dc}(\mathcal{B}))$, then:

$$\|p_{h,k} - P\|_{L^2(0,T,H^1(\mathcal{B}))} \leq \frac{C}{\alpha_T - \theta(k)}(\theta(k) + h + k)$$

If we replace P by its interpolant of $P_{h,k}$, and if P is C^4:

$$\|p_{h,k} - P_{h,k}\|_{L^2(0,T,H^1(\mathcal{B}))} \leq \frac{C}{\alpha_T - \theta(k)}(\theta(k) + h^2 + k^2 \log(T/K))$$

Test case

We built two test cases such that:

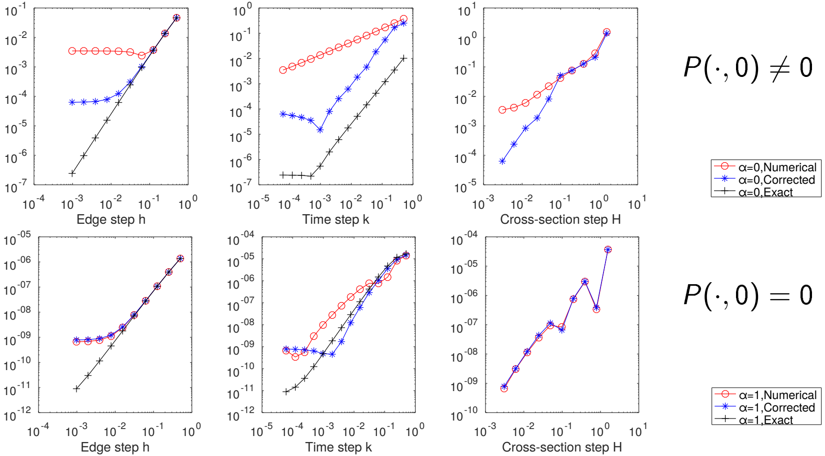
- The kernel for the cross-section is known (disk).
- One with a smooth pressure which vanishes and initial time and the other one such that is nonzero at initial time.
- $\Psi_\ell, f^{(e_i)}$ are smooth.

We used the scheme with:

- the exact kernel $K_q^{(\sigma)} = \frac{1}{k}\int_{t_q}^{t_{q+1}} K^{(\sigma)}(t)\mathrm{d}t$.
- a numerical approximation obtained with finite $\mathbb{P}^2$-elements and BDF2 integrator in the cross-section
- the same numerical approximation, corrected with an asympotic expansion of $K^{(\sigma)}$ for small times.

(See Éric Canon's talk for the details)

Numerical order

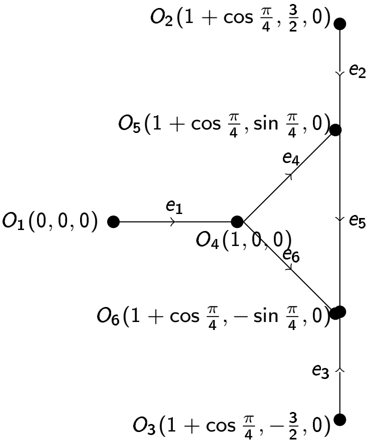


ℓ^∞-error on $\frac{\partial P}{\partial x^{(e)}}$ curves. On each graph, only one parameter varies, the two others are set by default to $h = 2^{-10}$, $k = 0.1 \cdot 2^{-14}$, $H = \pi 2^{-10}$.

Numerical order

		Numerical approximation	Corrected approximation	Exact
	h	2	2	2
$\beta = 0, P(\cdot, 0) \neq 0$	k	0.5	~ 1.4	$\frac{3}{2}$
$\beta = 1, P(\cdot, 0) = 0$	k	~ 1.6	~ 1.8	2
$\beta = 0, P(\cdot, 0) \neq 0$	H	1	~ 1.6	
$\beta = 1, P(\cdot, 0) = 0$	H	~ 1.7	~ 1.7	

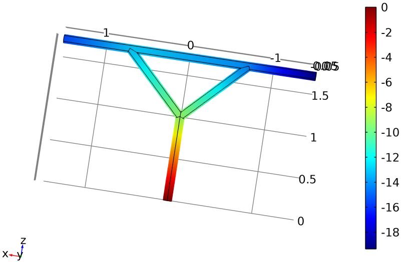
Test-case for the comparison with full Navier-Stokes equation



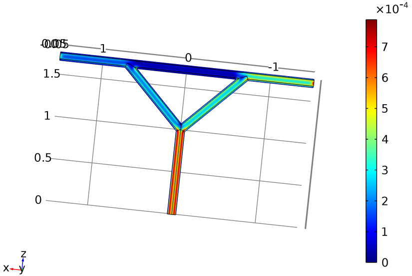
Let Ω^ε be the interior of $\{M \in \mathbb{R}^3 | \exists i, O \in e_i, OM \perp e_i, \|OM\| < \varepsilon\}$.
Let us take the following boundary conditions:

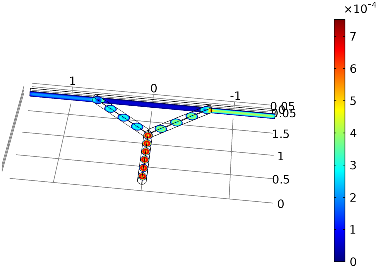
- $p = 0$ and $v(M,t)$ colinear to e_1 at the beginning O_1 of e_1.
- $v(M,t) = -\frac{e_2}{|e_2|} v_0 \sin\left(\varepsilon^{-2} 40t\right) \left(1 - 4\varepsilon^{-2}\|O_2 M\|^2\right)$ at the beginning O_2 of e_2. The flux through this tube is then $\frac{1}{8}\pi v_0 \varepsilon^2 \sin\left(\varepsilon^{-2} 40t\right)$;
- $v(M,t) = 2\frac{e_3}{|e_3|} v_0 \sin\left(\varepsilon^{-2} 40t\right) \left(1 - 4\varepsilon^{-2}\|O_3 M\|^2\right)$ at the beginning O_3 of e_3. The flux through this tube is then $\frac{\pi}{4} v_0 \varepsilon^2 \sin\left(\varepsilon^{-2} 40t\right)$;
- $v = 0$ on the rest of the boundary.

Pressure

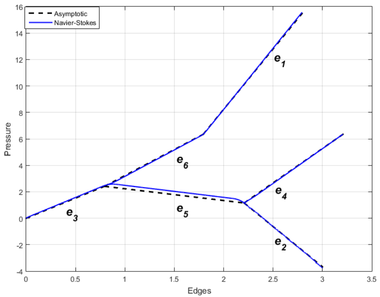


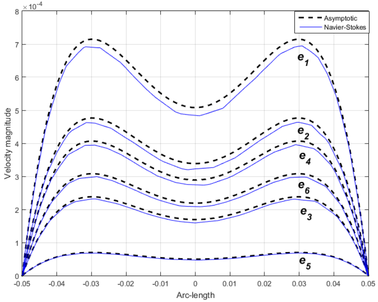
Velocity magnitude





Comparison between Navier-Stokes and the asymptotic model





Comparison between the asymptotic model (dashed lines) and the Navier-Stokes numerical solution (blue lines) for the multiply connected geometry when $T = 0.0875\varepsilon^2$, $\varepsilon = 0.1$. On the left, the pressure along tubes. On the right, the velocity magnitude across the middle of the six tubes with respect to the distance to the axis of the tube.

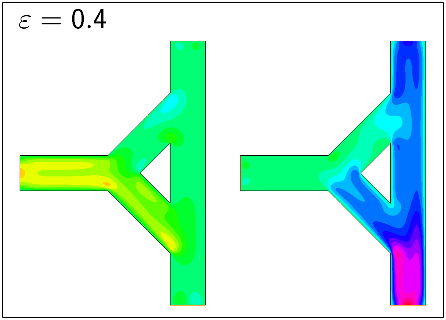
Comparison between the 3D Navier-Stokes numerical solution and the asymptotic model. $T = 0.875\varepsilon^2$.

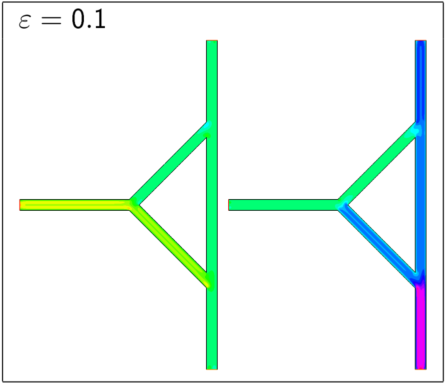
p^ϵ, P is the pressure on the graph for the NS numerical solution on the graph and for the asymptotic model.
q^ε is the orthogonal projection of p^ε on functions affine on each edge.
$\Phi_j^\varepsilon, \Phi_j$ is the flux accross the j-th tube according to Navier-Stokes numerical solution and the asymptotic model,

ε	0.2	0.1	0.05	0.025
$\frac{\|P-p^\varepsilon\|_{L^2(0,T,H^1(\mathcal{B}))}}{\|p^\varepsilon\|_{L^2(0,T,H^1(\mathcal{B}))}}$	0.144626	0.103521	0.080028	0.062730
$\frac{\|P-q^\varepsilon\|_{L^2(0,T,H^1(\mathcal{B}))}}{\|q^\varepsilon\|_{L^2(0,T,H^1(\mathcal{B}))}}$	0.036639	0.021016	0.031650	0.028022
$\frac{\left\|(\Phi_j-\Phi_j^\varepsilon)_j\right\|_{L^2(\{1,\ldots,M\}\times[0,T])}}{\left\|(\Phi_j^\varepsilon)_j\right\|_{L^2(\{1,\ldots,M\}\times[0,T])}}$	0.035878	0.030196	0.056603	0.053661

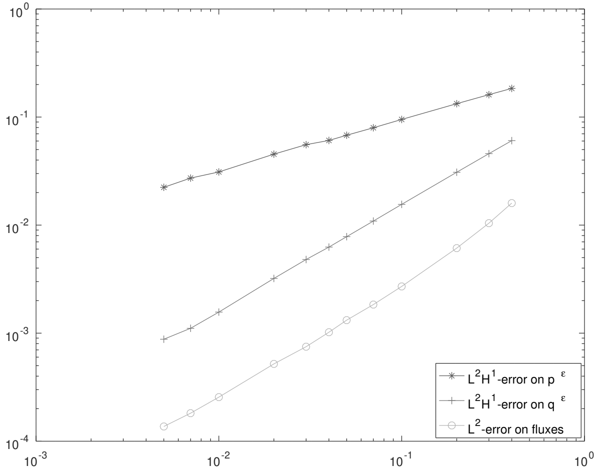
Remark: The Navier-Stokes simulation accuracy decreases when $\varepsilon \to 0$ because we were limited by computationnal cost.

Two-dimensionnal case





Comparison between the 2D Navier-Stokes and the asymptotic model



Conclusion

We got:

- Fast approximation of the asymptotic model.
- Good agreement with full Navier-Stokes equations.

Next talk by Éric Canon: Accurate approximation of the kernel K.

Thank you for your attention!

- Frédéric Chardard, Éric Canon, Grigori Panasenko, Olga Stikoniene.
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Test case

We consider the case of a single tube ($M = 1$) of length 1 with two extremities $O_1 = (\hat{0}, 0), O_2 = (\hat{0}, 1)$ ($N_1 = N = 2$). Let the cross-section of the tube be $\sigma = \{x \in \mathbb{R}^2; \|x\|_2 < 1, \}$.

Let us take $P((\hat{0}, x_3^{(e)}), t) = p(x_3^{(e)}, t) = \exp\left((1-t)x_3^{(e)} - \frac{\beta}{t}\right)$ where $\beta \in \{0, 1\}$. When $\beta = 1$, P and all its time derivatives are zero when $t \to 0$.

Then, the flow at the left extremity O_1 of the pipe is given by:

$$\Psi_1(t) = -\int_0^t K^{(\sigma)}(s)(1-(t-s))\exp\left(-\frac{\beta}{t-s}\right)\mathrm{d}s.$$

At the right extremity O_2 of the pipe, it is given by:

$$\Psi_2(t) = -\int_0^t K^{(\sigma)}(s)(1-(t-s))\exp\left(1-\frac{\beta}{t-s}\right)\mathrm{d}s.$$

The force applied along the pipe is:

$$F((\hat{0}, x_3^{(e)}), t) = -\int_0^t K^{(\sigma)}(s)(1-(t-s))^2\exp\left((1-(t-s))x_3^{(e)} - \frac{\beta}{t-s}\right)\mathrm{d}s$$