Configurations from strong deficient difference sets

Marién Abreu

Dipartimento di Matematica, Informatica ed Economia
Università degli Studi della Basilicata

joint work with
M. Funk, V. Krčadinac and D. Labbate

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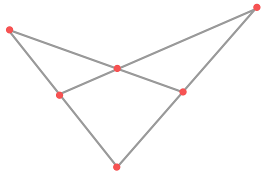
M.A., M. Funk, V. Krčadinac, D. Labbate,
Strongly regular configurations,
preprint, 2021. https://arxiv.org/abs/2104.04880

(v_r, b_k)–Configuration

Incident structure C consisting of:

v	points,
b	lines,
k	points on any line,
r	lines through any point,
NO pair of points belongs to two	distinct lines of C.

Example



$(6_2, 4_3)$

Graphs from configurations

Given a (v_k, b_r)-configuration C we can build graphs:

Point graph:

vertices	$=$ points of C
two vertices adjacent	if they belong to some common line of C.

And dually, we can define the

Line graph:

vertices	$=$ lines of C
two vertices adjacent	if they intersect at some point of C.

Strongly Regular Graphs

A graph is *strongly regular* with parameters $SRG(n, d, \lambda, \mu)$
if it has n vertices,
all of degree d,

such that
every pair of distinct vertices has λ common neighbours if they are adjacent,

and μ common neighbours if they are not adjacent.

Configurations and Strongly Regular Graphs

Some, but not all, point and line graphs of configurations are strongly regular.

Question

For which configurations are the point and line graphs both strongly regular?

When this happens we will say that the we have a

Strongly regular configuration

For more watch the talk *Strongly regular configurations* by Vedran Krčadinac (previous).

Deficient Difference Sets

Group G of order v

a *subset* D of size k
is a **deficient difference set** if the left differences

$d_1^{-1}d_2$ are all distinct for $d_1, d_2 \in D$, $d_1 \neq d_2$
equivalently the right differences are all distinct.

The *development* $\operatorname{dev} D = \{gD \mid g \in G\}$
considered as the lines of a configurations whose points are the elements of G gives rise to a

Symmetric (v_k) configuration which has G as an automorphism group acting regular on its points and lines.

Strong Deficient Difference Sets

A **Strong Deficient Difference Set** for $(v_k;\lambda,\mu)$ is a deficient difference set D of size k of a group G with v elements with the property that given the set of differences

$$\begin{aligned} \Delta(D) &= \{d_1^{-1}d_2 \mid d_1, d_2 \in D, d_1 \neq d_2\} \text{ and the numbers} \\ n(x) &= |\Delta(D) \cap x\Delta(D)| \text{ for } x \in G \setminus \{1\} \end{aligned}$$

it holds that

$$\begin{aligned} n(x) = \lambda &\quad \text{for every } x \in \Delta(D) \text{ and} \\ n(x) = \mu &\quad \text{for every } x \notin \Delta(D). \end{aligned}$$

We denote these sets with the acronym **SDDS**

Strongly Regular Configurations from SDDS

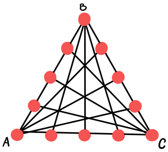
Theorem 1 (M.A., M. Funk, V. Krčadinac, D. Labbate, 2021 [1])

Let G be a group and $D \subseteq G$ a strong deficient difference set for $(v_k; \lambda, \mu)$.
Then, $(G, \operatorname{dev} D)$ is a strongly regular $(v_k; \lambda, \mu)$ configuration with G as an automorphism group acting regularly on the points and lines.
Conversely, any strongly regular $(v_k; \lambda, \mu)$ configuration with an automorphism group G acting regularly on the points and lines can be obtained from a $SDDS$ in G.

From Vedran Krčadinac's talk

Theorem 2 (M.A., M. Funk, V. Krčadinac, D. Labbate, 2021 [1])

Let $\mathcal{P}$ be a projective plane of order $n \geqslant 5$ and A, B, C be three non-collinear points. By deleting all points on the lines AB, AC, BC and all lines through the points A, B, C, there remains a strongly regular $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, and $\mu = (n-3)(n-4)$. This configuration is not an (semi) partial geometry.



In the Desarguesian projective plane $PG(2,q)$

All triangles {A,B,C} are equivalent the theorem gives just one strongly regular configuration up to isomorphism, which is self-dual.Choose $A = (0,0)$ and B, C on the “line at infinity”.

Let $G = \mathbb{F}_q^* \times \mathbb{F}_q^*$

Points of the configuration are pairs (x,y) with $x, y \in \mathbb{F}_q^*$

Lines are sets of points satisfying $y = ax + b$, $a, b \in \mathbb{F}_q^*$

The set $D = \{(x, x+1) \mid x \in \mathbb{F}_q^* \setminus \{-1\}\}$

is a SDDS for $(v_k; \lambda, \mu)$ generating the above lines. With parameters

$v = (q-1)^2$,

$k = q-2$,

$\lambda = (q-4)^2 + 1$, and

$\mu = (q-3)(q-4)$.

The full automorphism group of the configuration is

$((\mathbb{F}_q^* \times \mathbb{F}_q^*) : \mathrm{Aut}(\mathbb{F}_q)) : S_3$,

where $\mathrm{Aut}(\mathbb{F}_q)$ are the field automorphisms, and

For $n = 9$

Plane	$\lvert\mathrm{Aut}\rvert$	#Cf	Plane	$\lvert\mathrm{Aut}\rvert$	#Cf
$PG(2,9)$	768	1	Hughes	144	1
Hall	768	1		48	1
	96	2		32	1
	12	2		18	1
	6	1		12	3
Dual Hall	768	1		6	4
	96	2		4	3
	12	2		2	1
	6	1		1	1

Table: Distribution of $(64_7; 26, 30)$ configurations by order of full automorphism group.

In Hall's Plane of order 9 and its dual

the two $(64_7; 26, 30)$ configurations with full automorphism groups of order 768 arise from the group $G = Q_8 \times Q_8$ where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group with usual multiplication (e.g. $i^2 = j^2 = k^2 = -1$, $ij = k$)
The sets $SDDS$

$$D_1 = \{(1,1), (i,-k), (j,k), (k,-j), (-i,j), (-j,i), (-k,-i)\}$$

and

$$D_2 = \{(1,1), (i,-k), (j,j), (k,-j), (-i,-i), (-j,i), (-k,k)\}$$

The first gives the configuration constructed from the Hall plane, coordinatized by the quaternionic near-field, when two of the points $\{A, B, C\}$ are chosen on the translation line and
the second gives the dual configuration, obtained in the dual Hall plane when one of the points of the triangle is the translation point.

Exhaustive search up to $v = 200$

We performed an exhaustive computer search for strong deficient difference sets* in groups of order $v \leqslant 200$, using the GAP library of small groups.

Besides the two $(64_7; 26, 30)$
we found four other examples not corresponding to Theorem 2.

The configurations constructed from these SDDS's have flag-transitive automorphism groups.

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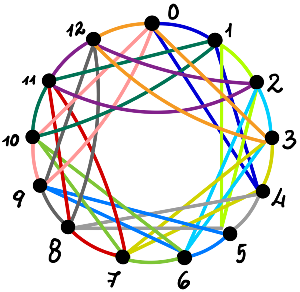
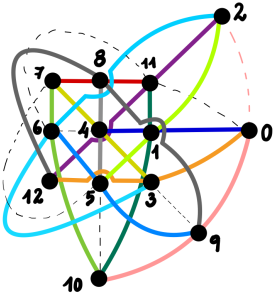
* with parameters corresponding to proper and primitive strongly regular configurations
(see [1] or Krčadinac's talk for definitions)

Example 1: In the cyclic group $\mathbb{Z}_{13}$

there is one SDDS fixed by the multiplier 3: $D = \{7, 8, 11\}$.
The development has full automorphism group $\mathbb{Z}_{13} : \mathbb{Z}_3$ acting flag-transitively.



Point graph of $(13_3; 2, 3)$



$(13_3; 2, 3)$ embedded in $PG(2, 3)$

Only cyclic strongly regular configuration we found.
Emenddable in the projective plane of order 3 by adding a point to every line.

Example 2: SDDS's for $(96_5; 4, 4)$

In the groups $\mathbb{Z}_4 \times S_4$, $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4) : \mathbb{Z}_2$, $D_8 \times A_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_4$ there are $SDDS$

One SDDS in $\mathbb{Z}_4 \times S_4$ is

$$D = \{(0, id), (1, (1,4)(2,3)), (1, (1,3,4,2)), (1, (1,4,3)), (2, (1,2,4))\}.$$

The developments are all isomorphic and give one self-dual configuration.

The full automorphism group is $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) : A_6) : \mathbb{Z}_2$ of order 11520 and acts flag-transitively.

The graphs from an SDDS $(96_5; 4, 4)$

The associated graph is a $SRG(96, 20, 4, 4)$.

Many SRGs with these parameters are known, see e.g.

[3](2003) by Brouwer, Koolen and Klin
and
[8](2006) by Golemac, Mandić, Vučičić

In [3] this graph is called K'' and the corresponding configuration is mentioned.

The graph with these parameters with largest automorphism group of order 138240 is the point graph of the generalized quadrangle $pg(5, 3, 1)$.

Example 3: SDDS's for $(120_8; 28, 24)$

For example, in the symmetric group S_5,

$D = \{id, (1,2,5,3,4), (1,3,4,2,5), (1,5,3,2,4), (1,4)(2,3,5), (1,4,5,2), (1,2,4), (1,2,5)\}$

Up to isomorphism one self-dual strongly regular configuration arises.

The full automorphism group is isomorphic to the alternating group A_8 of size 20160 and acts flag-transitively.

The graphs from an SDDS $(120_8; 28, 24)$

This $(120_8; 28, 24)$ configuration was constructed in [2](1997) by Brouwer, Haemers and Tonchev

by embedding the $pg(7, 8, 4)$ of
[5](1980) De Clerck, Dye and Thas and [6](1981) Cohen
into a Steiner 2-$(120, 8, 1)$ design.

The 135 lines of the $pg(7, 8, 4)$ and the 120 lines of the configuration cover every pair of the 120 points exactly once and form a design.

The point graphs of the $pg(7, 8, 4)$ and the $(120_8; 28, 24)$ configuration are complementary with parameters $SRG(120, 63, 30, 36)$ and $SRG(120, 56, 28, 24)$, respectively.

The graphs from an SDDS $(120_8; 28, 24)$

In [5] De Clerck, Dye and Thas the $pg(7,8,4)$ is part of an infinite family constructed from the hyperbolic quadric in $PG(4n-1,2)$.

The family is denoted by $PQ^+(2n-1,2)$ and has parameters $pg(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2})$.

These parameters fit a hypothetical $(v_k; \lambda, \mu)$ configuration with $v = 2^{2n-1}(2^{2n}-1)$, $k = 2^{2n-1}$, $\lambda = 2^{2n-2}(2^{2n-1}-1)$, and $\mu = 2^{2n-1}(2^{2n-2}-1)$ to make a 2-$(v,k,1)$ design.

But in [2] the paper by Brouwer, Haemers, Tonchev it was proved that it is not possible to make such a 2-design for $n > 2$.

Non-isomorphic partial geometries with the same parameters that could possibly be embedded in Steiner 2-designs were constructed in [9](1997) by Mathon and Street; and in [4](2000) by De Clerck and Delanote.

Example 4: SDDS's for $(155_7; 17, 9)$

Represent the group $G = \mathbb{Z}_{31} : \mathbb{Z}_5$ as permutations of $\mathbb{Z}_{31}$ generated by

$$f : x \mapsto x + 1 \pmod{31} \text{ and } g : x \mapsto 2x \pmod{31}$$

Then, $D = \{id, f^{12}g^4, f^{15}g, f^{18}, f^{20}g^2, f^{26}g^3, f^{30}\}$ is a SDDS.

One self-dual strongly regular configuration arises, isomorphic to the semipartial geometry $LP(4, 2)$.

The full automorphism group $P\Gamma L(5, 2)$ is of order 9999360 and acts flag-transitively.

$LP(4,2)^{\pi}$, $LP(4,2)_{\pi'}$ and $LP(4,2)^{\pi}_{\pi'}$ do not arise from SDDS

Vedran Krčadinac showed in his talk that each semipartial geometry $LP(4,q)$ gives rise to at least four strongly regular configurations by polarity transformations $LP(4,q)^{\pi}$, $LP(4,q)_{\pi'}$, $LP(4,q)^{\pi}_{\pi'}$ of which only the original one is a semipartial geometry.

However already for $q=2$, the configurations obtained from $LP(4,2)$ by polarity transformations cannot be constructed from SDDS because their full automorphism groups are not transitive.

In particular, the dual pair $LP(4,2)^{\pi}$ and $LP(4,2)_{\pi'}$ have full automorphism groups of order 322560 isomorphic to $(\mathbb{Z}_2)^4 : P\Gamma L(4,2)$. The group acts in orbits of size 35, 120 on the points and 15, 140 on the lines of $LP(4,2)^{\pi}$, and vice versa for $LP(4,2)_{\pi'}$.

The self-dual configuration $LP(4,2)^{\pi}_{\pi'}$ has full automorphism group of order 20160 isomorphic to $P\Gamma L(4,2)$ acting in orbits of size 15, 35, 105.

Example 5: strongly regular configurations not from SDDS

Vedran Krčadinac also showed in his talk that there are at least four non-isomorphic $(63_6; 13, 15)$ configurations. But they do not arise from $SDDS$ because their automorphism groups do not act regularly.

Two of them are self-dual with full automorphism groups $PSU(3,3) : \mathbb{Z}_2$ of order 12096 acting flag-transitively (related to generalize hexagon $GH(2,2)$ - see [7])

This is a (63_3) configuration with point and line graphs of girth 12 and diameter 6. The graphs are distance regular, but not strongly regular.

And there is a dual pair with full automorphism groups $(SL(2,3) : \mathbb{Z}_4) : \mathbb{Z}_2$ of order 192 acting in orbits of size 1, 6, 24, 32, found computationally, by prescribing automorphism groups.

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Thank You