Maximal cliques in strongly regular graphs

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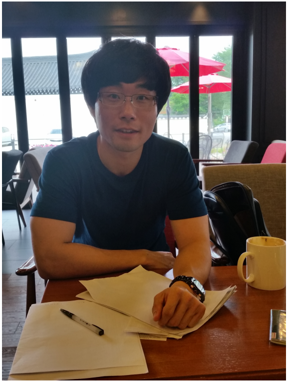
Plan

- Bounds for cliques in graphs
- **Main result**: forbidden interval for maximal cliques
- Nonexistence of strongly regular graphs

GG, J.H. Koolen, J. Park, European J. Combin. (to appear)



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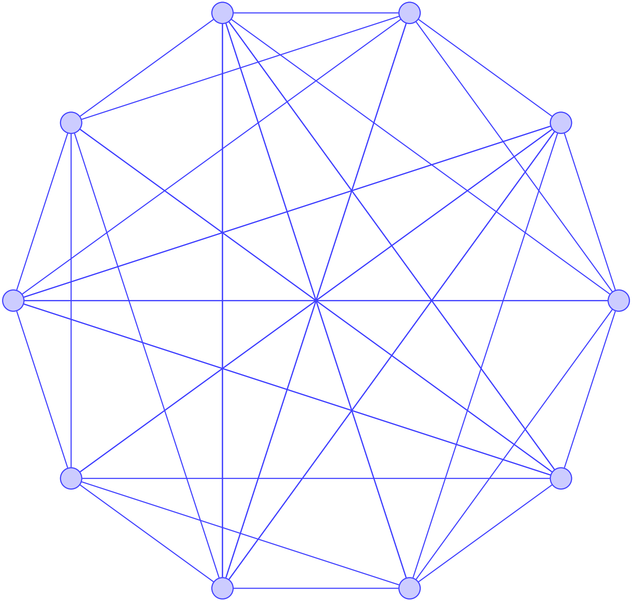

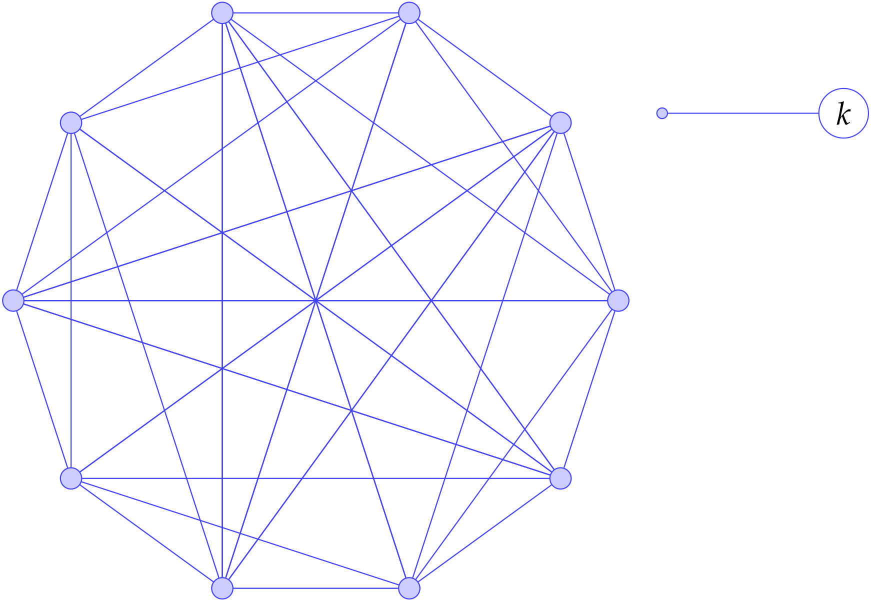
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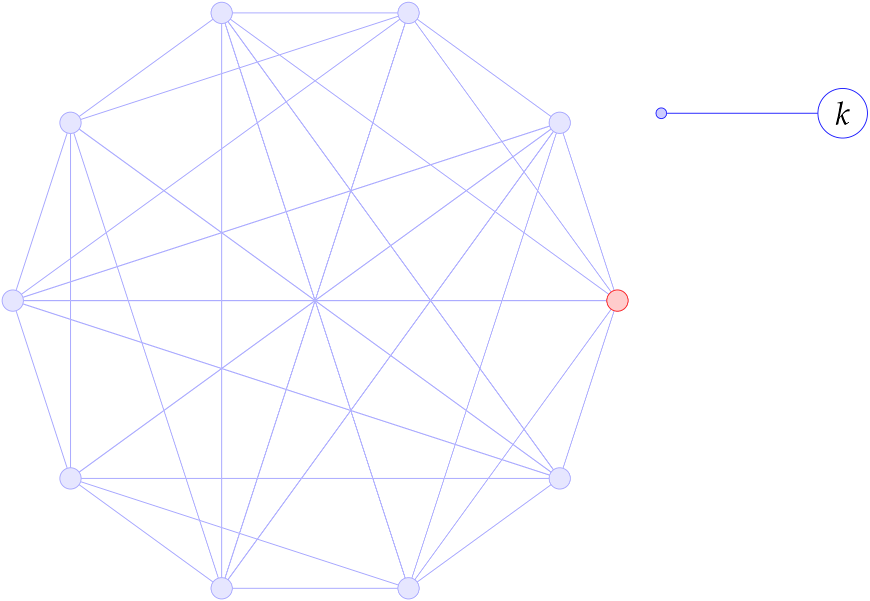
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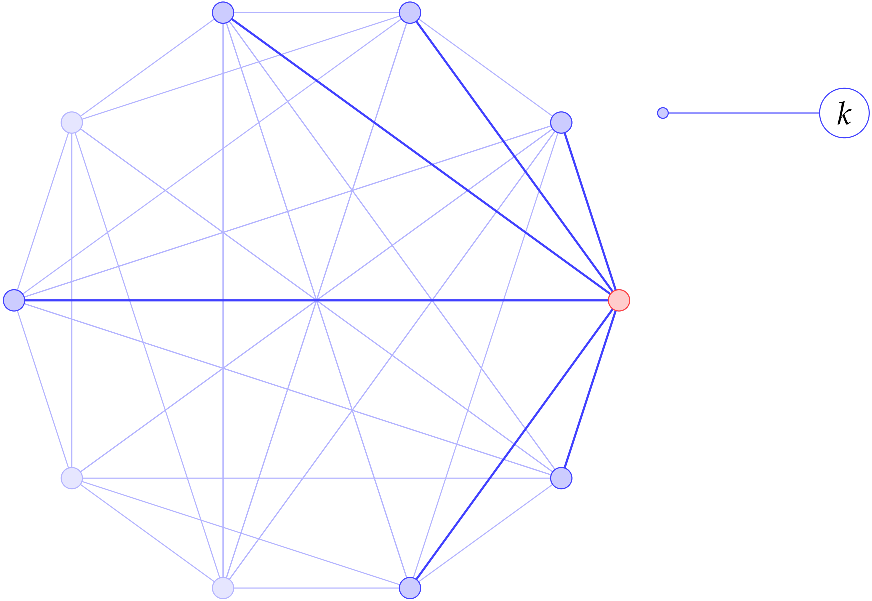
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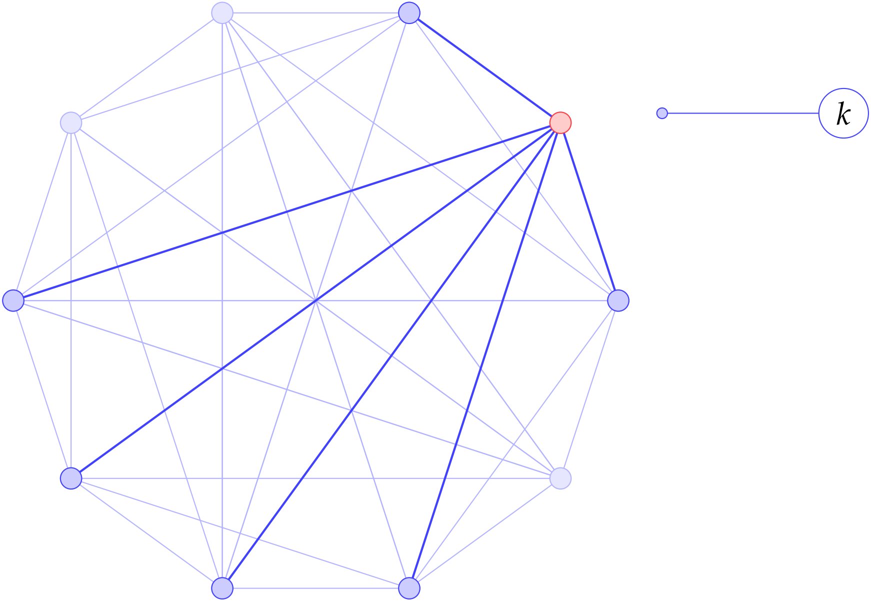
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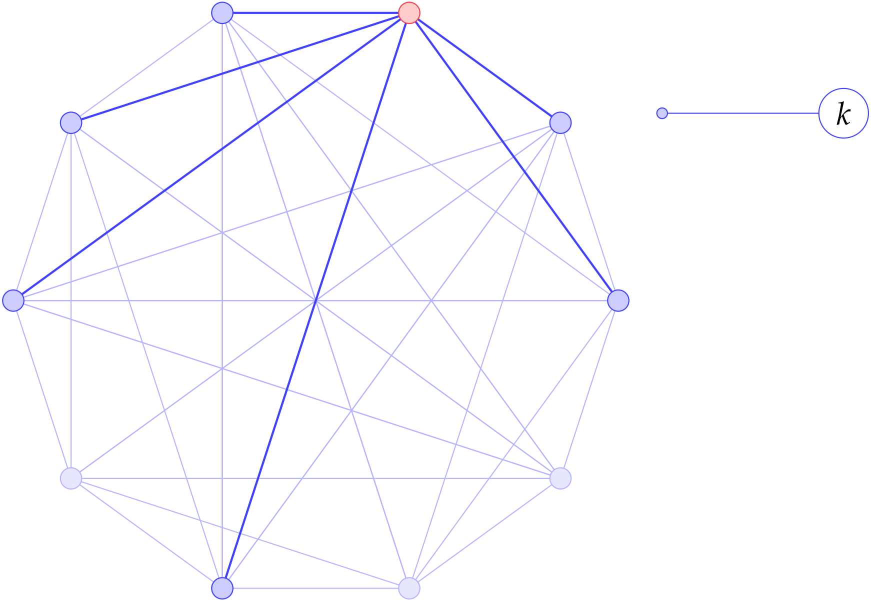
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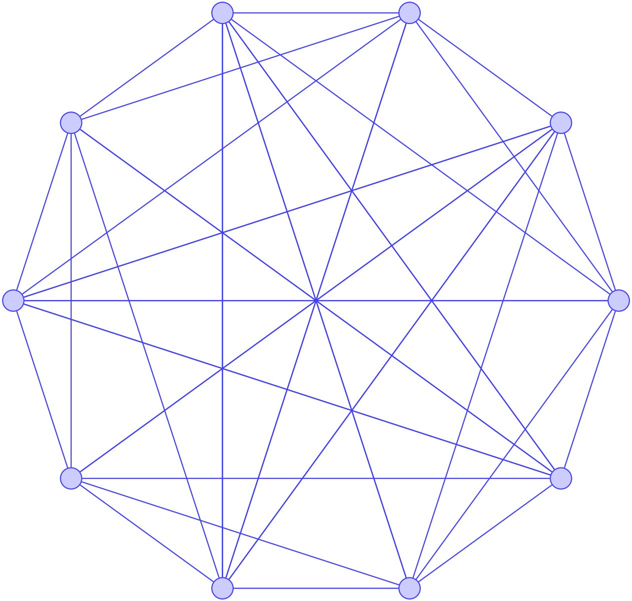

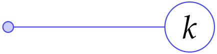

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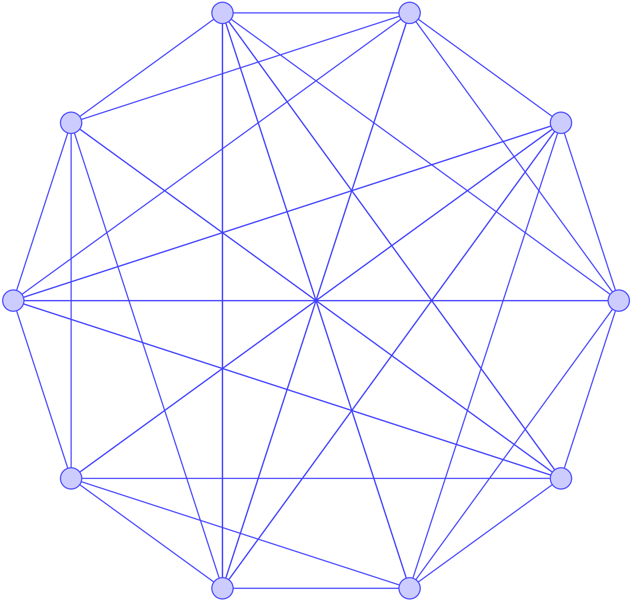

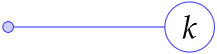

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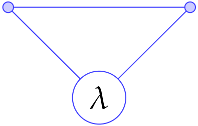

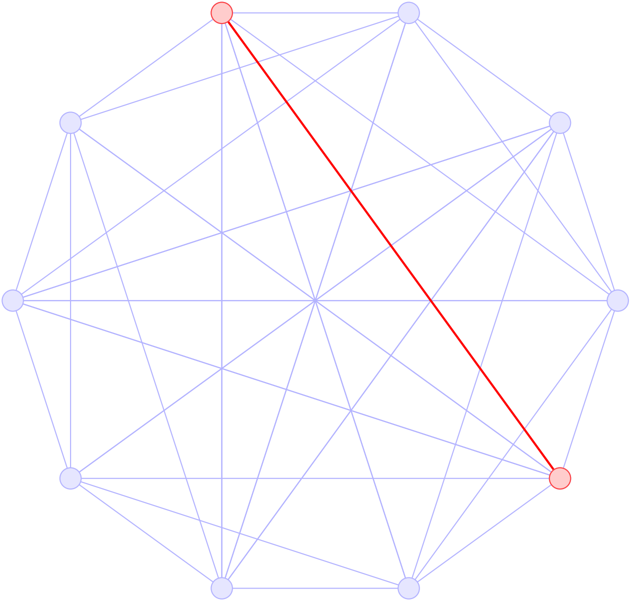

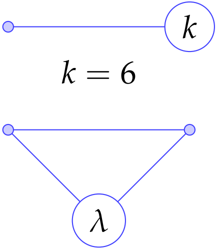
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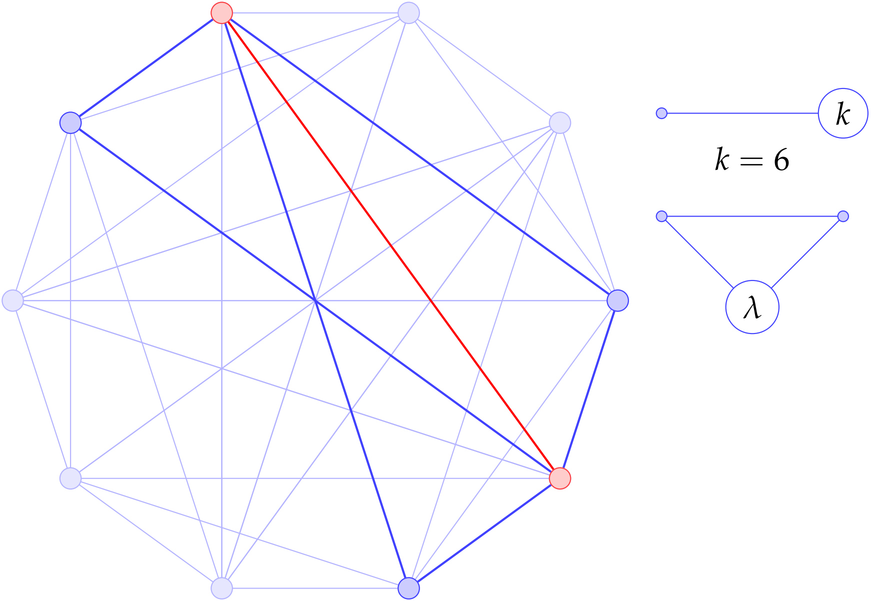
λ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k = 6
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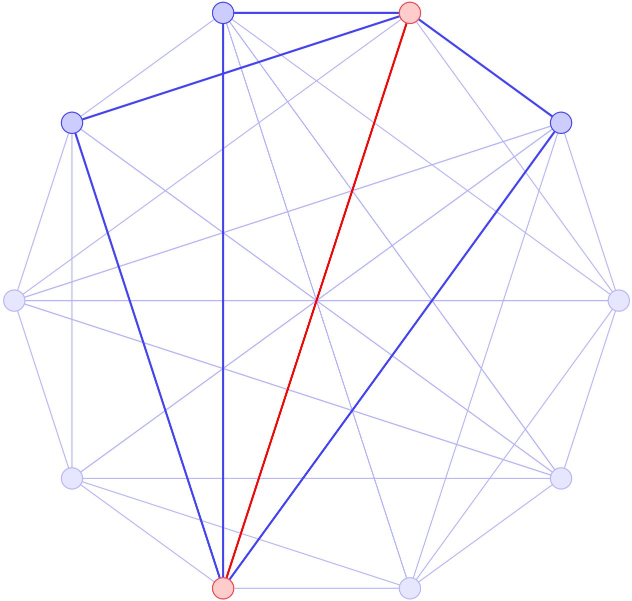
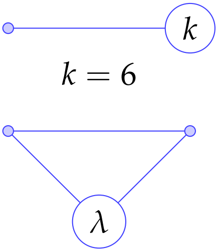
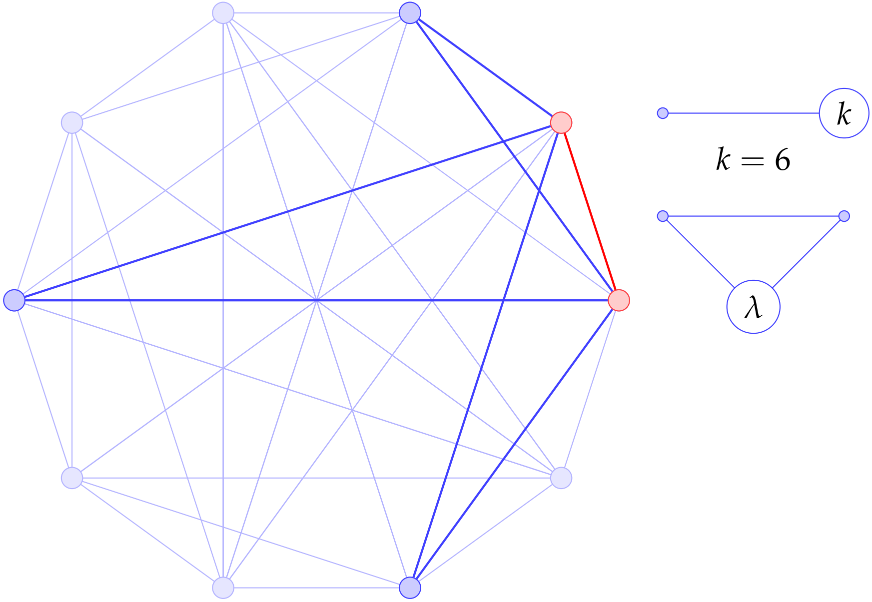
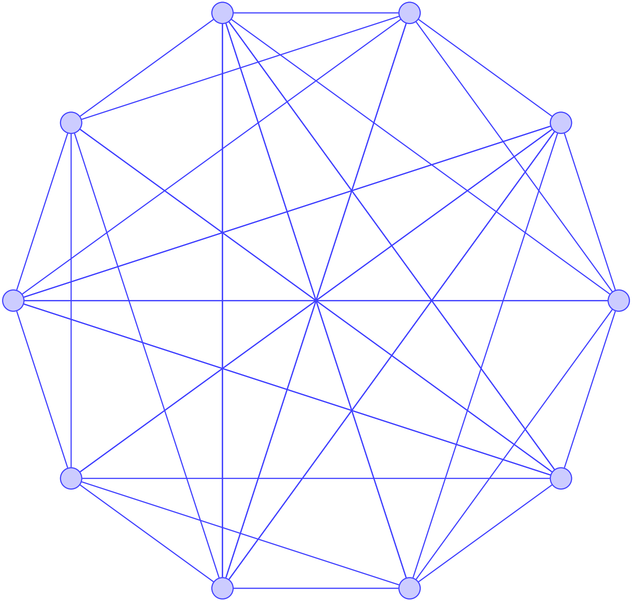
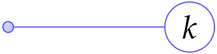



k
$k = 6$
λ



k
$k = 6$
λ





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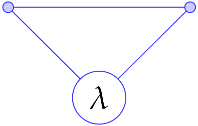
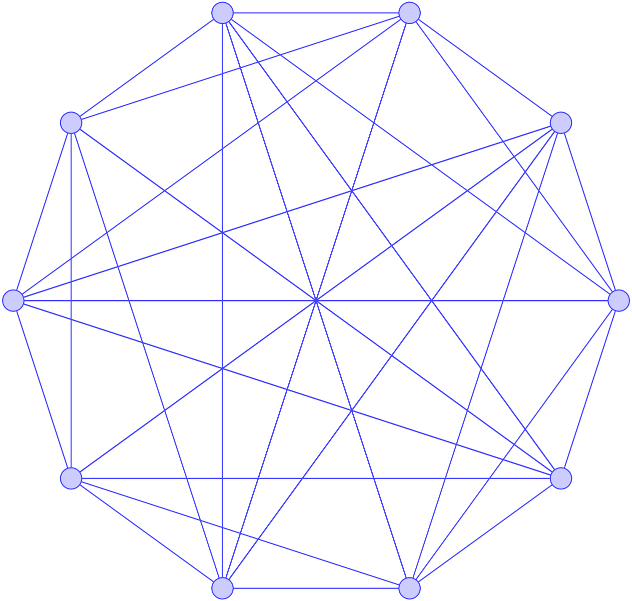
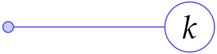


$\lambda = 3$





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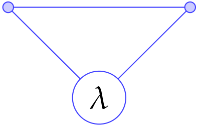
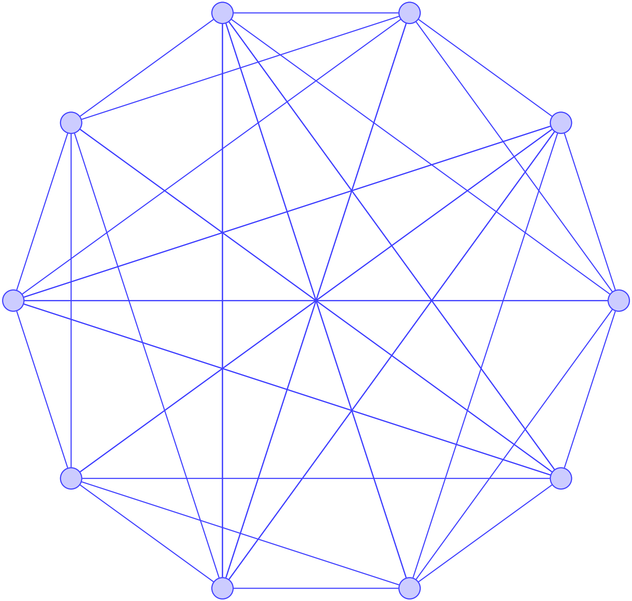
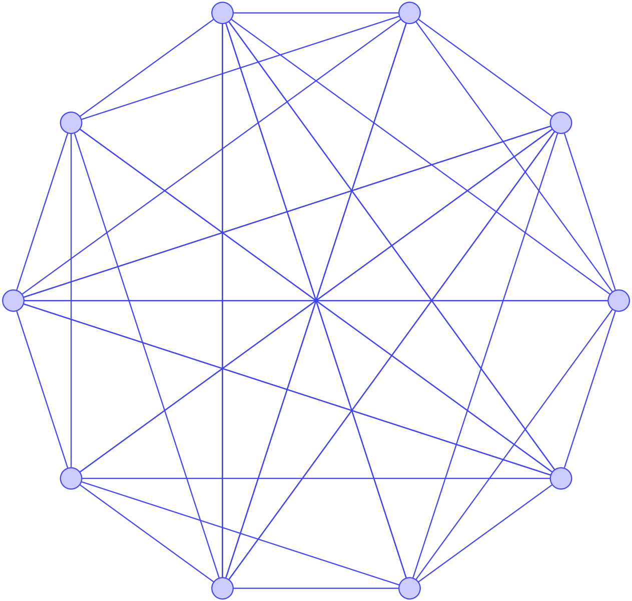


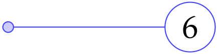
$\lambda = 3$

edge-regular

$\mathrm{erg}(10, 6, 3)$



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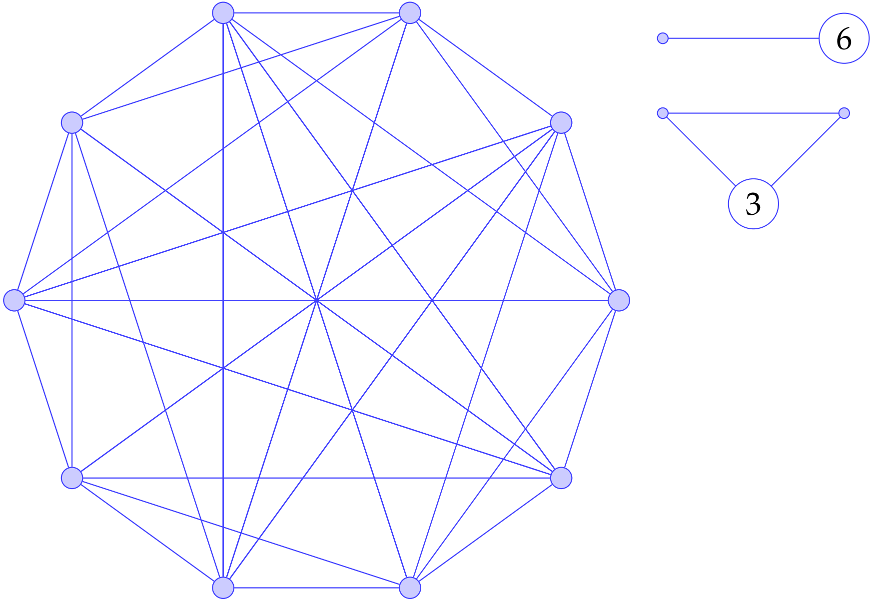
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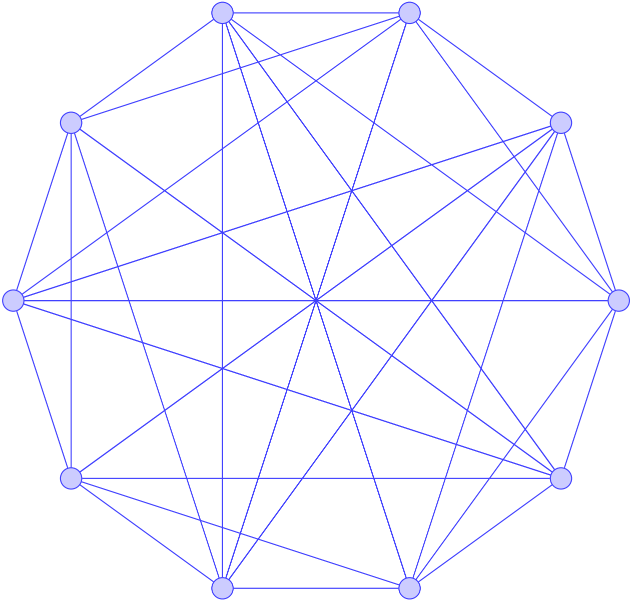

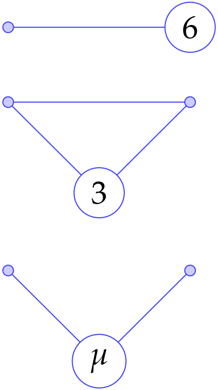
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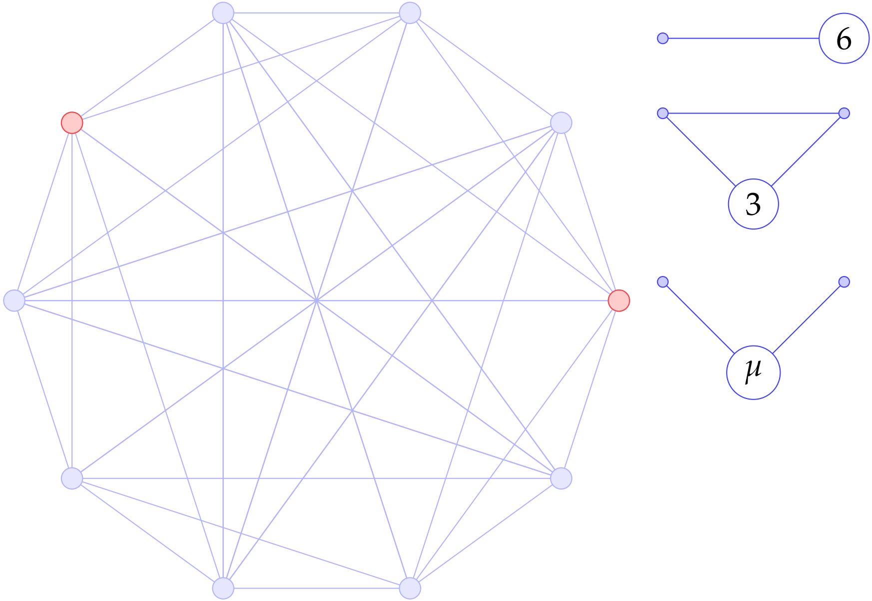
μ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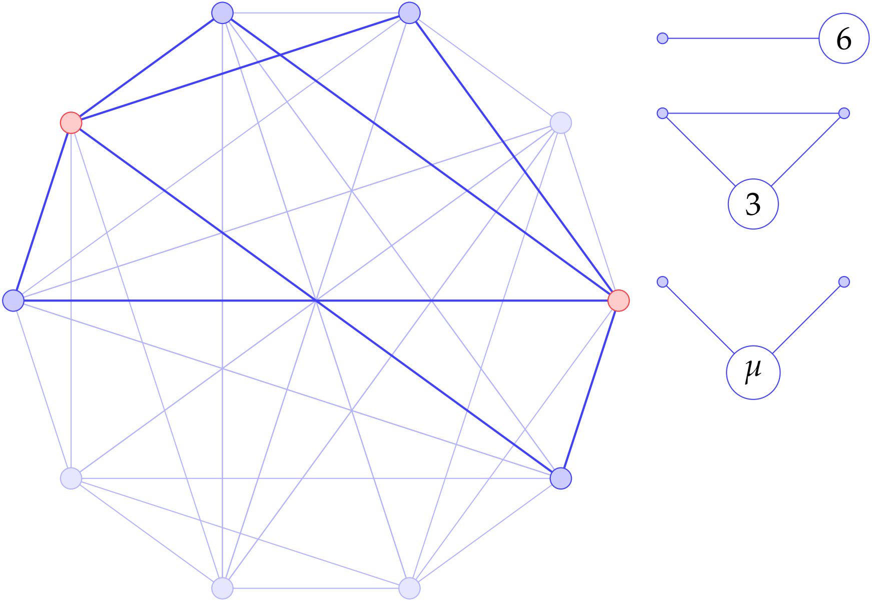
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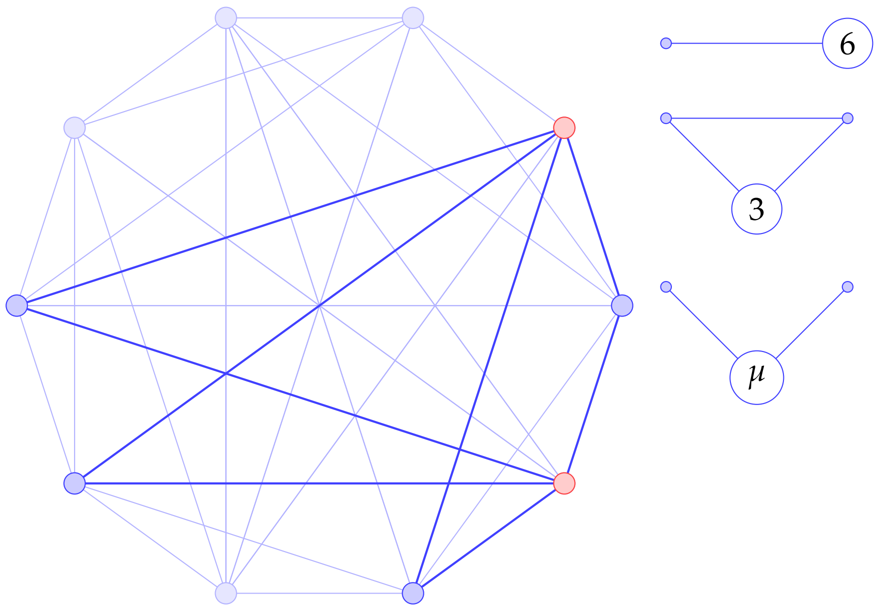
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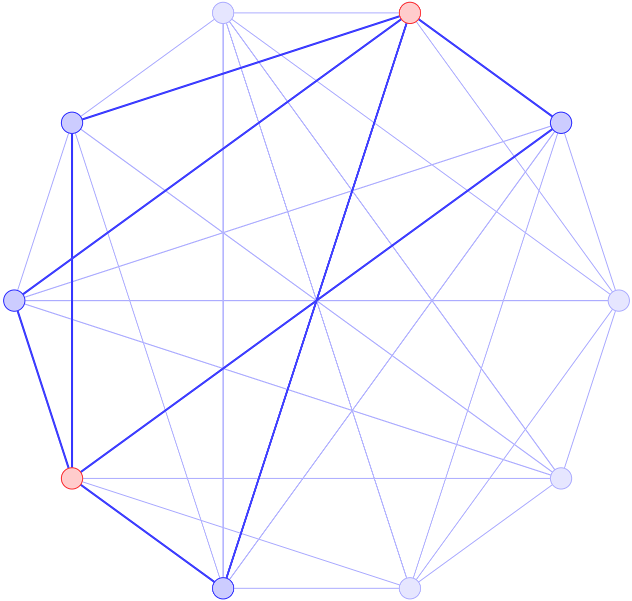

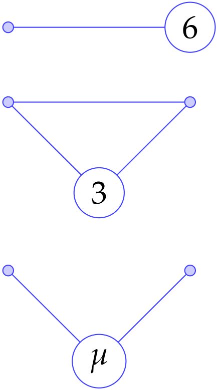
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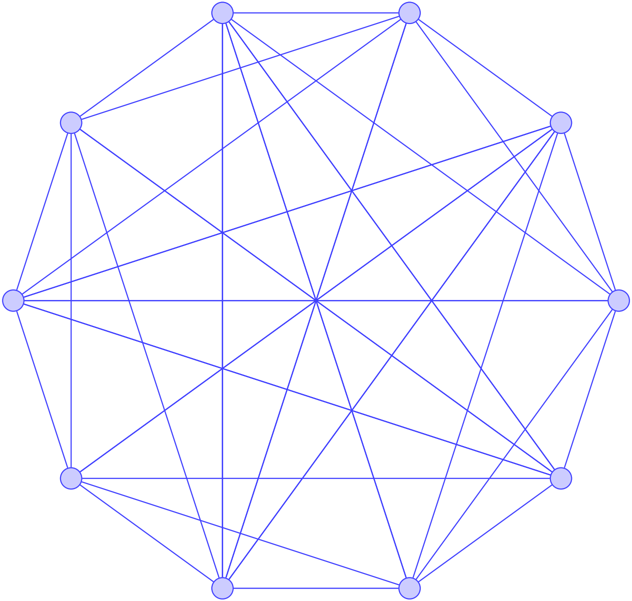
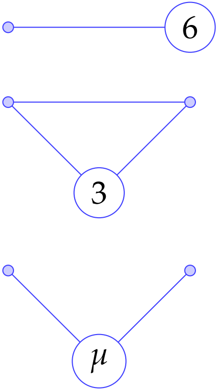
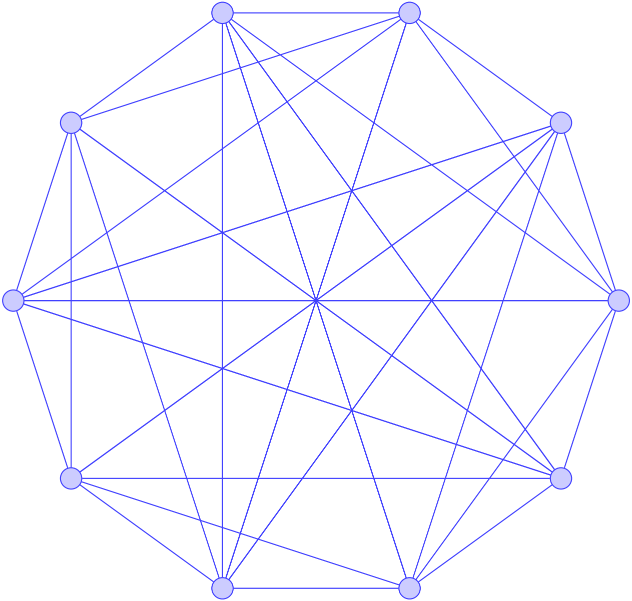
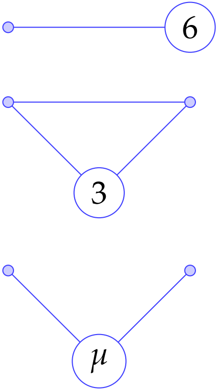




$\mu = 4$

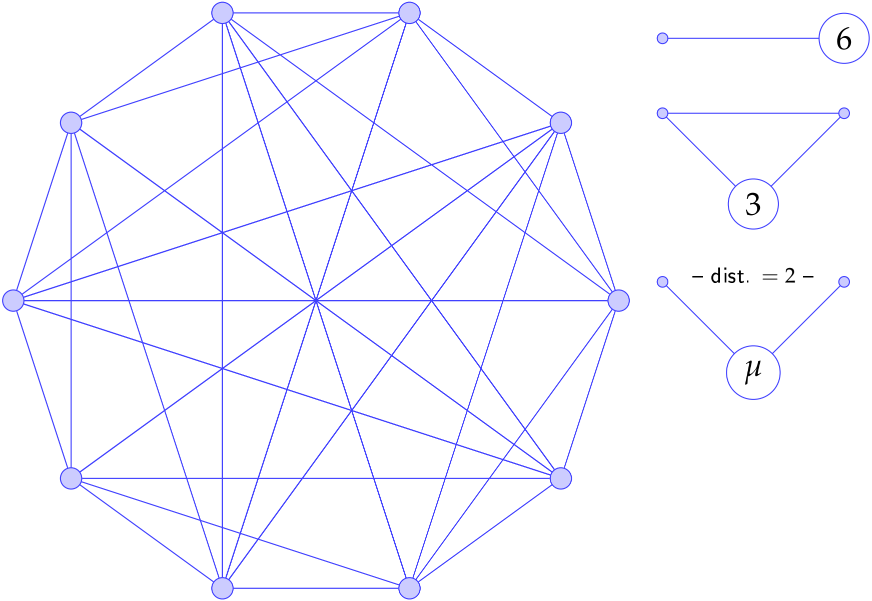


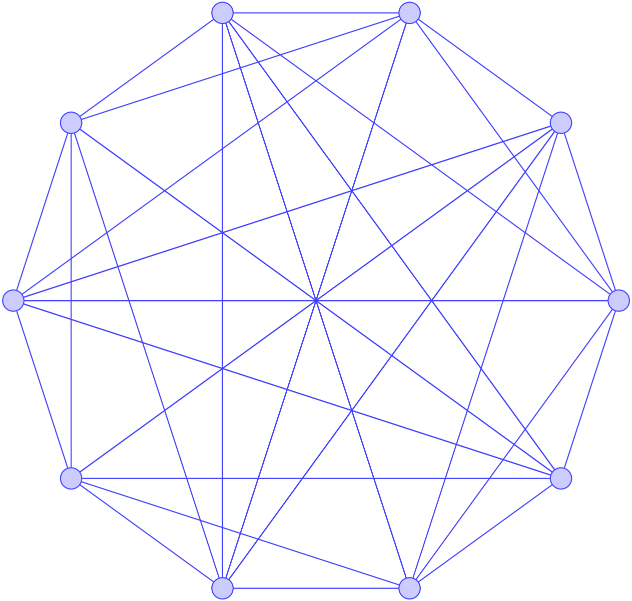


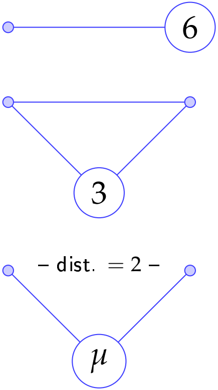
$\mu = 4$

strongly regular

srg$(10, 6, 3, 4)$


6
3
– dist. = 2 –
μ





$\mu = 4$

amply regular

$\mathrm{arg}(10, 6, 3, 4)$

(Co-)clique bounds

Let Γ be a graph with v vertices and smallest eigenvalue $-m$.

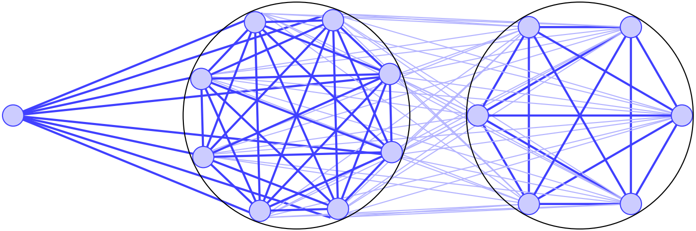
- $\alpha(\Gamma)$: independence number of Γ;
- $\omega(\Gamma)$: clique number of Γ;
- n_+: number of positive eigenvalues of Γ;
- n_-: number of negative eigenvalues of Γ.

Cvetković: $\alpha(\Gamma) \leqslant \min\{v - n_+, v - n_-\}$.

Hoffman: If Γ is k-regular then $\alpha(\Gamma) \leqslant \frac{vm}{k+m}$.

Delsarte: If Γ is an $\mathrm{srg}(v,k,\lambda,\mu)$ then $\omega(\Gamma) \leqslant 1 + \frac{k}{m}$.

Main tool: $H(a,t)$

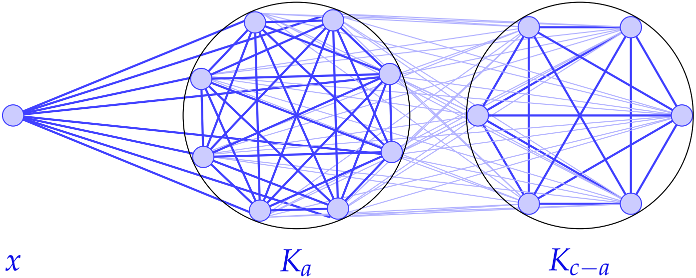


Quotient matrix of $H(a,t)$: $$Q = \begin{bmatrix} 0 & a & 0 \\ 1 & a-1 & t \\ 0 & a & t-1 \end{bmatrix}.$$

Observation: If graph Γ has smallest eigenvalue $-m$ and contains $H(a,t)$ then

$$(a - m(m-1))(t - (m-1)^2) \leqslant (m(m-1))^2.$$

Main tool: $H(a,t)$



Quotient matrix of $H(a, c-a)$: $Q = \begin{bmatrix} 0 & a & 0 \\ 1 & a-1 & c-a \\ 0 & a & c-a-1 \end{bmatrix}.$

Observation: If graph Γ has smallest eigenvalue $-m$ and contains $H(a, c-a)$ then

$$(a - m(m-1))(c - a - (m-1)^2) \leqslant (m(m-1))^2.$$

Maximal clique polynomial

Let Γ be an $\mathrm{erg}(v,k,\lambda)$ with smallest eigenvalue $-m$. Define $M_\Gamma(c)$ as the polynomial

$$M_\Gamma(c) := ((c+m-3)(k-c+1) - 2(c-1)(\lambda-c+2))^2 - (k-c+1)^2(c+m-1)(c-(m-1)(4m-1))$$

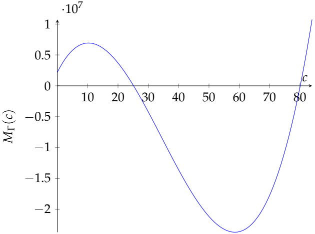
Under certain assumptions involving the order c of a maximal clique, $M_\Gamma(c)$ must be nonnegative.

GG, Koolen, Park (2021): Let Γ be an $\mathrm{arg}(v,k,\lambda,\mu)$ with smallest eigenvalue $-m$ such that $\mu > m(m-1)$. Suppose that Γ has a maximal clique C of order $c > \frac{\mu^2}{\mu-m(m-1)} - m + 1$. Then $M_\Gamma(c) \geqslant 0$.

Augmenting the Delsarte bound

Upper bound order of a clique of $\Gamma = \mathrm{srg}(1344, 221, 88, 26)$.

The smallest eigenvalue of Γ is $-m$ is -3.
$M_\Gamma(c) = 544c^3 - 56160c^2 + 980544c + 2200896$.



$M_\Gamma(c) < 0$ for $c \in [26, 80]$

$26 = \mu > m(m-1) = 6$

$\frac{\mu^2}{\mu - m(m-1)} - m + 1 = 31.8$

- If Γ has a maximal clique of order $c \geqslant 32$ then $c > 80$
- Delsarte: $\omega(\Gamma) \leqslant 74$
- Delsarte + Maximal Clique Polynomial: $\omega(\Gamma) \leqslant 31$

Lower bound for cliques in ARGs

Let Γ be a graph and x a vertex of Γ.
$\Gamma(x)$ denotes the set of neighbours of x.

Godsil (1993): Let Γ be an $\arg(v,k,\lambda,\mu)$, let x be a vertex of Γ, and let $\overline{C}$ be a coclique of order $\bar{c} \geqslant 2$ in $\Gamma(x)$. Then

$$\binom{\bar{c}}{2}(\mu - 1) \geqslant \bar{c}(\lambda + 1) - k$$

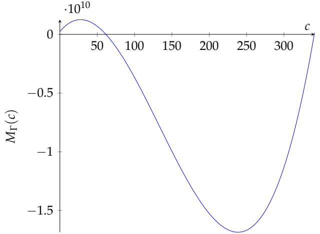
If $\binom{\bar{c}}{2}(\mu - 1) < \bar{c}(\lambda + 1) - k$, then cocliques in $\Gamma(x)$ have order $< \bar{c}$

GG, Koolen, Park (2021): Let Γ be an $\arg(v,k,\lambda,\mu)$. If $\binom{\bar{c}}{2}(\mu - 1) < \bar{c}(\lambda + 1) - k$ for some integer $\bar{c} \geqslant 2$, then Γ contains a clique of order at least $2 + \lambda - (\bar{c} - 2)(\mu - 1)$.

Nonexistence of strongly regular graphs

Nonexistence of $\Gamma = \mathrm{srg}(23276, 1330, 372, 58)$.

The smallest eigenvalue of Γ is $-m$ is -4.
$M_\Gamma(c) = 3864c^3 - 1542672c^2 + 76457832c + 243482976$.



$M_\Gamma(c) < 0$ for $c \in [62, 340]$

$58 = \mu > m(m-1) = 12$

$$\frac{\mu^2}{\mu - m(m-1)} - m + 1 = 70.130\ldots$$

- If Γ has a maximal clique of order $c \geqslant 71$ then $c > 340$
- Delsarte: $\omega(\Gamma) \leqslant 333$
- Delsarte + Maximal Clique Polynomial: $\omega(\Gamma) \leqslant 70$

Nonexistence of strongly regular graphs

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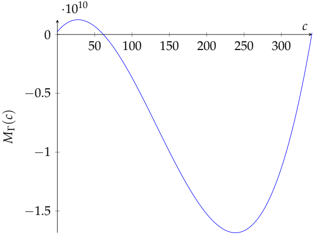

$M_\Gamma(c) < 0$ for $c \in [62, 340]$

$58 = \mu > m(m-1) = 12$

$\frac{\mu^2}{\mu - m(m-1)} - m + 1 = 70.130\ldots$

- Delsarte + Maximal Clique Polynomial: $\omega(\Gamma) \leqslant 70$
- $\binom{6}{2}(\mu - 1) < 6(\lambda + 1) - k \implies \Gamma$ has a clique of order $2 + \lambda - (6-2)(\mu - 1) = 146$.

More nonexistent parameters

$m = 4$:

v	k	λ	μ	forbidden range	Delsarte bound	guaranteed clique order
23276	1330	372	58	[71, 340]	333	146
25025	1426	399	62	[74, 368]	357	157
27455	1696	480	80	[92, 450]	425	166
38875	2046	569	82	[94, 539]	512	247

$m = 5$:

v	k	λ	μ	forbidden range	Delsarte bound	guaranteed clique order
133570	4365	960	115	[136, 885]	874	278
230958	5917	1276	122	[142, 1202]	1184	673
235586	6625	1440	150	[170, 1367]	1326	697
317628	7747	1666	152	[172, 1593]	1550	913
328560	8283	1786	168	[187, 1714]	1657	953
259000	7395	1610	170	[189, 1538]	1480	767
309016	8127	1758	172	[191, 1686]	1626	905
225885	7580	1675	205	[224, 1605]	1517	453
404587	10374	2241	214	[233, 2170]	2075	1178
314116	9675	2122	240	[258, 2052]	1936	929
485815	12040	2595	240	[258, 2524]	2409	1402

Infinitely many nonexistent parameters

GG, Koolen, Park (2021): Let $m \geqslant 4$ be an integer. Then there are no strongly regular graphs with the following parameters:

$$v = 1 + k + k(k - \lambda - 1)/\mu,$$

$$k = (m+1)(m(2-\mu) + 2\lambda)/2 + 1,$$

$$\lambda = \frac{(m-3)^5 + 15(m-3)^4 + 91(m-3)^3 + 283(m-3)^2 + 452(m-3) + 296}{2},$$

$$\mu = (m-3)^3 + 10(m-3)^2 + 33(m-3) + 38.$$

m	v	k	λ	μ	forbidden range	Delsarte bound	clique order
4	38875	2046	569	82	[94, 539]	512	247
5	317628	7747	1666	152	[172, 1593]	1550	913
6	1756209	23108	4057	254	[284, 3915]	3852	2541
7	7404736	58305	8660	394	[436, 8417]	8330	5911
⋮							⋮