Heffter Arrays

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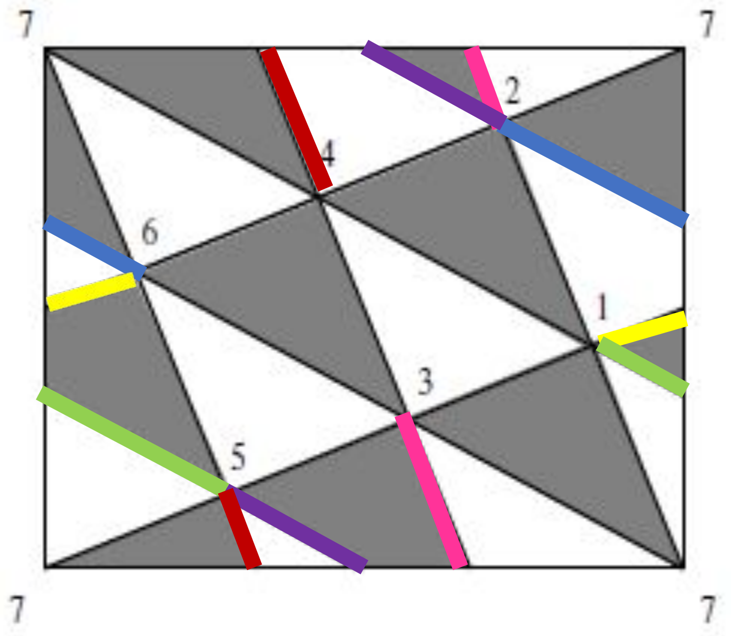
Joint work with K. Burrage, D. Donovan, N. Cavenagh

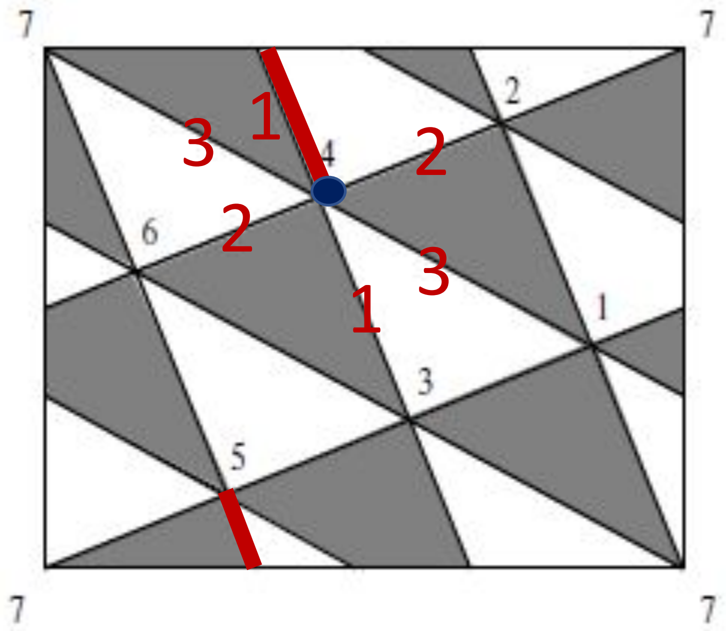
Biembedding cycle systems on surfaces



Dark triangles:
156
267
371
412
523
634
745

White triangles:
126
237
341
452
563
674
715

Biembedding cycle systems on surfaces

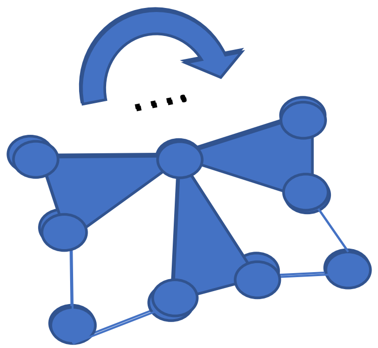


Dark triangles:	White triangles:
156	126
267	237
371	341
412	452
523	563
634	674
745	715

Face 2-colorable biembeddings of cycle systems on surfaces

We will have difference sets for 3-cycles:
{1, 7, -8}
{-2, -4, 6}
{5, 9, 11}
{-3, -10, -12}

We will have difference sets for 4-cycles:
{1, -2, -10, 11}
{-4, 7, 9, -12}
{-3, 5, 6, -8}



1	-2	11	-10
7	-4	9	-12
-8	6	5	-3

H(3,4;4,3)

K_n should be decomposable into 3-cycles and 4-cycles so $n \equiv 1 \bmod 24$.
Let $n=25$
We have the differences:

1 2 3 …. 12

Difference set: {1, -2, -10, 11}
Blocks:
0 1 24 14
1 2 0 15
2 3 1 16
3 4 2 17
.
.
.

Heffter Arrays

- A *Heffter array* H($m,n;s,t$) is an m x n *matrix with nonzero entries from* $\mathbb{Z}_{2ms+1}$ *such that :*

  - *each row contains s filled cells and each column contains t filled cells,*
  - *the elements in every row and column sum to 0 in* $\mathbb{Z}_{2ms+1}$*, and*
  - *for every* $x \in \mathbb{Z}_{2ms+1}/\{0\}$*, either x or –x appears in the array.*

Observe that in an Heffter array ms=nt

If $m = n$ we denote it by $H(m;t)$ (it is a square array).

If the elements in each row and column sum to 0 in $\mathbb{Z}$, it is called an *integer* Heffter array.

H(6,12;8,4)

		-1	2	29	-30			-25	26	5	-6
		3	-4	-31	32			27	-28	-7	8
9	-10			-13	14	33	-34			-37	38
-11	12			15	-16	-35	36			39	-40
-17	18	21	-22			-41	42	45	-46		
19	-20	-23	24			43	-44	-47	48		

Tight Heffter array (all entries filled) H(4,6;6,4)

1	-2	3	-4	11	-9
-7	8	-12	10	-5	6
-13	14	-15	16	-23	21
19	-20	24	-22	17	-18

Square integer Heffter array H(6;4)

		-1	2	5	-6
		3	-4	-7	8
9	-10			-13	14
-11	12			15	-16
-17	18	21	-22		
19	-20	-23	24		

What do we know ?

- Theorem: (Archdeacon, Cavenagh,Dinitz, Donovan, Wanless, Yazıcı, 2019)

  There exist an $H(n,k)$ if and only if $3 \leq k \leq n$.

- A $H(n,m;m;n)$ (tight Heffter arrays) exist if $m \geq 3$ and $n \geq 3$.

H(6,12;8,4)

		-1	2	29	-30			-25	26	5	-6
		3	-4	-31	32			27	-28	-7	8
9	-10			-13	14	33	-34			-37	38
-11	12			15	-16	-35	36			39	-40
-17	18	21	-22			-41	42	45	-46		
19	-20	-23	24			43	-44	-47	48		

N=2*48+1=97
Difference set {-1,2,29,-30,-25,26,5,-6}
Base block: (0, 96, 1, 30, 0 ,72, 1, 6)
not a 8-cycle
But if you reorder the differences
{-1,2,-25,26,5,29,-6,-30}
Base block: (0, 96, 1, 73, 2, 7, 36, 30)

Base block: (0, 96, 1, 73, 2, 7, 36, 30)

(1,0,2,74,3,8,37,31)
(2,1,3,75,4,9,38,32)
.
.
.

Simple and globally simple Heffter arrays

- Given a row r of a Heffter array H(m,n;s;t), if there exists a cyclic ordering $\boldsymbol{\Phi}_r=(a_1,a_2,\ldots,a_s)$ of entries of row r such that for i∈{1,..,s}

$$\alpha^i = \sum_{j=0}^{i} a_j \ (mod\ 2ms+1)$$

are all ***distinct,*** we say that $\boldsymbol{\Phi}_r$ is simple. If all rows and columns of a Heffter array is simple that we say that the array is simple.

- If the natural ordering (ordering from left to right for rows and top to bottom for columns) is simple for every row and every column then the array is called ***globally simple***.

What do we know about globally simple Heffter arrays?

- Theorem: (Costa,Morini,Pasotti,Pellegrini, 2018) There exists globally simple integer Heffter arrays H(n;k) when $3 \leq k \leq 10$.

- Note: Integer Heffter array exists iff $3 \leq k \leq n$ and $nk \equiv 0$ or $3 \bmod 4$.

- We will prove that there exists a globally simple Heffter array H(n,k)
  1. $k \equiv 0 \bmod 4$ and $n \geq k$.
  2. $k \equiv 3 \bmod 4$ and $n \equiv 1 \bmod 4$ and $n \geq k$.
  3. $k \equiv 3 \bmod 4$ and $n \equiv 0 \bmod 4$ and $n \gg k$.

- $k \equiv 1 \bmod 4$ and $k \equiv 2 \bmod 4$ still open

Simple Heffter arrays and orthogonal cycle decompositions

- A s-cycle system S is called orthogonal to a t-cycle system T if the cycles in S have at most one edge in common with each cycle in T.
- Simple Heffter arrays imply orthogonal cycle systems so:
- **Theorem**: There exists a pair of orthogonal decompositions of K_{2nk+1} into cycles of length k where $n \geq k$ and $nk \equiv 0$ or $3 \bmod 4$ if
  1. $k \in \{5,10\}$ and $n \equiv 3 \bmod 4$
  2. $k = 6$ and $n \equiv 2 \bmod 4$
  3. $k \equiv 0 \bmod 4$
  4. $k \equiv 3 \bmod 4$ and $n \equiv 1 \bmod 4$
  5. $k \equiv 3 \bmod 4$ and $n \equiv 0 \bmod 4$ and $n \gg k$.

Globally simple H(n,4p)

- Assume p≥3 is odd and k=4p.
- **Example: L**et n=29 and k=28=4x7.
- Consider the sequence :

1 -2 -25 24 5 -6 -21 20 9 -10 -17 16 13 -14 -12 11 18 -19 -8 7 22 -23 -4 3 26 -27 -15 28

Partial sums:

1 -1 -26 -2 3 -3 -24 -4 5 -5 -22 -6 7 -7 -19 -8 10 -9 -17 -10 12 -11 -15 -12 14 -13 -28 0

mod 8:

1 27 2 26 3 25 4 24 5 23 6 22 7 21 9 20 10 19 11 18 12 17 13 16 14 15 0

We see all possible equivalence classes mod 28 as partial sums

Construction k=4p, p odd
H(17,12)

Partial sums	x mod 12
1	1
-1=11	-2
-10=2	-9
10	8
3	5
-3=9	-6
5	-4
8	3
6	10
-5=7	-11
0	-7
	0

x

1						96	−91	−119	118	135	−136	−162	161	188	−189	−2
−14	13						108	−103	−131	130	147	−148	−174	173	200	−201
−9	−26	25						120	−115	−143	142	159	−160	−186	185	8
20	−21	−38	37						132	−127	−155	154	171	−172	−198	197
5	32	−33	−50	49						144	−139	−167	166	183	−184	−6
−18	17	44	−45	−62	61						156	−151	−179	178	195	−196
−4	−30	29	56	−57	−74	73						168	−163	−191	190	3
15	−16	−42	41	68	−69	−86	85						180	−175	−203	202
10	27	−28	−54	53	80	−81	−98	97						192	−187	−11
−23	22	39	−40	−66	65	92	−93	−110	109						204	−199
−7	−35	34	51	−52	−78	77	104	−105	−122	121						12
24	−19	−47	46	63	−64	−90	89	116	−117	−134	133					
	36	−31	−59	58	75	−76	−102	101	128	−129	−146	145				
		48	−43	−71	70	87	−88	−114	113	140	−141	−158	157			
			60	−55	−83	82	99	−100	−126	125	152	−153	−170	169		
				72	−67	−95	94	111	−112	−138	137	164	−165	−182	181	
					84	−79	−107	106	123	−124	−150	149	176	−177	−194	193

Construction
k≡3 mod 4

1	-2					-29	30
32	3	-4					-31
-17	18	5	-6				
	-19	20	7	-8			
		-21	22	9	-10		
			-23	24	11	-12	
				-25	26	13	-14
-16					-27	28	15

Construction k≡3 mod 4

1	-2				-27	28	
	3	-4				-29	30
32		5	-6				-31
-17	18		7	-8			
	-19	20		9	-10		
		-21	22		11	-12	
			-23	24		13	-14
-16				-25	26		15

+ 5

6	-7				-32	33	
	8	-9				-34	35
37		10	-11				-36
-22	23		12	-13			
	-24	25		14	-15		
		-26	27		16	-17	
			-28	29		18	-19
-21				-30	31		20

Shiftable Heffter array

Construction k≡3 mod 4

1	-2	53	-54	-85	-37	38			86			
	3	-4	55	-56	-101	-39	40			102		
		5	-6	57	-58	-91	-41	42			92	
			7	-8	59	-60	-81	-43	44			82
98				9	-10	61	-62	-97	-45	46		
	88				11	-12	63	-64	-87	-47	48	
		104				13	-14	65	-66	-103	-49	50
52			94				15	-16	67	-68	-93	-51
-27	28			84				17	-18	69	-70	-83
-99	-29	30			100				19	-20	71	-72
-74	-89	-31	32			90				21	-22	73
75	-76	-79	-33	34			80				23	-24
-26	77	-78	-95	-35	36	38		96				25

H(13,8)

Support shifted globally simple Heffter arrays

- The array A of size n is defined to be a support shifted simple integer Heffter array $H(n; 4p, \gamma)$ if it satisfies the following properties:
  1. Every row and every column of A has 4p filled cells.
  2. $s(A) = \{\gamma n + 1, . . . , (4p + \gamma)n\}$.
  3. Elements in every row and every column sum to 0.
  4. Partial sums are distinct in each row and each column of A modulo $2(4p + \gamma)n + 1$.

H(17;12,3)

85					252		−105	104	−169	168	−253	−212	213	−148	149	−84
−52	53					224		−103	102	−167	166	−225	−214	215	−150	151
153	−54	55					230		−101	100	−165	164	−231	−216	217	−152
−120	121	−56	57					236		−99	98	−163	162	−237	−218	219
221	−122	123	−58	59					242		−97	96	−161	160	−243	−220
−188	189	−124	125	−60	61					248		−95	94	−159	158	−249
−255	−190	191	−126	127	−62	63					254		−93	92	−157	156
154	−227	−192	193	−128	129	−64	65					226		−91	90	−155
−187	186	−233	−194	195	−130	131	−66	67					232		−89	88
86	−185	184	−239	−196	197	−132	133	−68	69					238		−87
−119	118	−183	182	−245	−198	199	−134	135	−70	71					244	
	−117	116	−181	180	−251	−200	201	−136	137	−72	73					250
222		−115	114	−179	178	−223	−202	203	−138	139	−74	75				
	228		−113	112	−177	176	−229	−204	205	−140	141	−76	77			
		234		−111	110	−175	174	−235	−206	207	−142	143	−78	79		
			240		−109	108	−173	172	−241	−208	209	−144	145	−80	81	
				246		−107	106	−171	170	−247	−210	211	−146	147	−82	83

H(n,3)

-8	18							-10
-19	-7	26						
	-11	-6	17					
		-20	-5	25				
			-12	-9	21			
				-16	3	13		
					-24	2	22	
						-15	1	14
27							-23	-4

L’

				-20		-5		25
27					-23		-4	
	21					-12		-9
-8		18					-10	
	3		13					-16
-19		-7		26				
	-24		2		22			
		-11		-6		17		
			-15		1		14	

L

$L(2(i+1)+b, 2j) = L'(i,j)$ where $b=1$

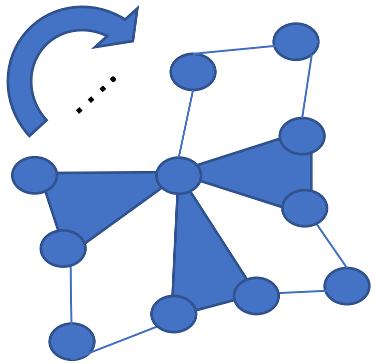
H(17;15)

85		−33	244	13		20	−105	104	−169	168	−245	−212	213	−148	149	−84
−52	53		−24	240	−4		28	−103	102	−167	166	−241	−214	215	−150	151
153	−54	55		−49	236	12		37	−101	100	−165	164	−237	−216	217	−152
−120	121	−56	57		−41	232	−3		44	−99	98	−163	162	−233	−218	219
221	−122	123	−58	59		−32	228	11		21	−97	96	−161	160	−229	−220
−188	189	−124	125	−60	61		−25	224	−2		27	−95	94	−159	158	−225
−255	−190	191	−126	127	−62	63		−48	254	10		38	−93	92	−157	156
154	−251	−192	193	−128	129	−64	65		−42	250	−1		43	−91	90	−155
−187	186	−247	−194	195	−130	131	−66	67		−31	246	9		22	−89	88
86	−185	184	−243	−196	197	−132	133	−68	69		−26	242	8		20	−87
−119	118	−183	182	−239	−198	199	−134	135	−70	71		−47	238	17		30
35	−117	116	−181	180	−235	−200	201	−136	137	−72	73		−51	234	16	
	46	−115	114	−179	178	−231	−202	203	−138	139	−74	75		−39	230	−7
15		19	−113	112	−177	176	−227	−204	205	−140	141	−76	77		−34	226
222	−6		29	−111	110	−175	174	−223	−206	207	−142	143	−78	79		−23
−50	252	14		36	−109	108	−173	172	−253	−208	209	−144	145	−80	81	
	−40	248	−5		45	−107	106	−171	170	−249	−210	211	−146	147	−82	83

How it all started

- Theorem (Archdeacon 2014): Given a Heffter array H = H($m,n;s,t$) with compatible simple orderings ω_r on the rows and ω_c on the columns of H, there exists an orientable embedding of K_{2ms+1} such that every edge is on a face of size s and a face of size t, i.e. a biembedding of a s-cycle system and a t-cycle system.

Compatible orderings



- We say that ω_r and ω_c are <u>*compatible*</u> if the permutation $\omega_r \circ \omega_c$ is a single cyclic permutation.
- The composition of the row and column orderings gives the ordering of the faces around each vertex. <u>Compatibility</u> ensures there are no "pinch points".

Compatible orderings

- If there exist compatible orderings ω_r and ω_c for a Heffter array $H(m, n; s, t)$, then either:
  1. m, n, s and t are all odd;
  2. m is odd, n is even and s is even; or
  3. m is even, n is odd and t is even.
- When m=n, H(n,k) with a compatible ordering: **n is odd**.

  We also need to have **nk ≡0 or 3 mod 4** for integer arrays. Hence:
  1. n ≡ 1 mod 4 and k ≡ 3 mod 4
     we solved this with same sporadic exceptions

  2. n ≡ 3 mod 4 and k ≡ 1 mod 4 **STILL OPEN**

Compatible orderings

- (Dinitz, Mattern, 2017) There exists an $H(m, n; n, m)$ which admits both simple and compatible orderings,
  1. for all $n \geqslant 3$ when $m = 3$, and
  2. for all $3 \leqslant n \leqslant 100$ when $m = 5$.
- There exists an integer $H(n; k)$, $n \geqslant k$ and $nk \equiv 3 \pmod 4$, that admit both simple and compatible orderings, when
  1. (Archdecon, Dinitz, Donovan,2014) k is 3
  2. *(Dinitz, Wanless, 2018) k is 5*
  3. *(Costa, Morini, Pasotti, Pellegrini, 2018)* $k \in \{7, 9\}$

H(17;15)

α

85		-33	244	13		20	-105	104	-169	168	-24	-212	213	-148	149	-84
-52	53		-24	240	-4		28	-103	102	-167	166	-241	-214	215	-150	151
153	-54	55		-49	236	12		37	-101	100	-165	164	-237	-216	217	-152
-120	121	-56	57		-41	232	-3		44	-99	98	-163	162	-233	-218	219
221	-122	123	-58	59		-3	228	11		21	-97	96	-161	160	-229	-220
-188	189	-124	125	-60	61		-25	224	-2		27	-95	94	-159	158	-225
-255	-190	191	-126	127	-62	63		-48	254	10		38	-93	92	-157	156
154	-251	-192	193	-128	129	-64	65		-47	250	-1		43	-91	90	-155
-187	186	-247	-194	195	-130	131	-66	67		-3	246	9		22	-89	88
86	-185	184	-243	-196	197	-132	133	-68	69		-26	242	8		2	-87
-119	118	-183	182	-239	-198	199	-134	135	-70	71		-47	238	17		3
35	-117	116	-181	180	-235	-200	201	-136	137	-72	73		-51	234	16	
	46	-115	114	-179	178	-231	-202	203	-138	139	-74	75		-39	230	-7
15		19	-113	112	-177	176	-227	-204	205	-140	141	-76	77		-34	226
222	-6		29	-111	110	-175	174	-223	-206	207	-142	143	-78	79		-23
-50	252	14		36	-109	108	-173	172	-253	-208	209	-144	145	-80	81	
	-40	248	-5		45	-107	106	-171	170	-249	-210	211	-146	147	-82	83

Our Results

- If there exists α such that $2p + 2 \leqslant \alpha \leqslant n - 2 - 2p$, $\gcd(n, \alpha) = 1$, $\gcd(n, \alpha - 2p - 1) = 1$ and $\gcd(n, n-1 - \alpha - 2p) = 1$, then there exists a globally simple integer Heffter array $H(n; 4p + 3)$ with an ordering that is both simple and compatible.

- Let $n \equiv 1 \pmod 4$, $p > 0$, $n > k = 4p + 3$ and either:

  (a) n is prime;

  (b) $n = k + 2$; or

  (c) $n \geqslant 7(k + 1)/3$ and if $n \equiv 3 \pmod 6$ then $p \equiv 1 \pmod 3$.

Then there exists a globally simple integer Heffter array $H(n; k)$ with an ordering that is both simple and compatible. Furthermore, there exists a face 2-colorable embedding of K_{2nk+1} on an orientable surface such that the faces of each color are cycles of length k.

H(17;15)

85		−33	244	13		20	−105	104	−169	168	−245	−212	213	−148	149	−84
−52	53		−24	240	−4		28	−103	102	−167	166	−241	−214	215	−150	151
153	−54	55		−49	236	12		37	−101	100	−165	164	−237	−216	217	−152
−120	121	−56	57		−41	232	−3		44	−99	98	−163	162	−233	−218	219
221	−122	123	−58	59		−37	228	11		21	−97	96	−161	160	−229	−220
−188	189	−124	125	−60	61		−25	224	−2		27	−95	94	−159	158	−225
−255	−190	191	−126	127	−62	63		−48	254	10		38	−93	92	−157	156
154	−251	−192	193	−128	129	−64	65		−42	250	−1		43	−91	90	−155
−187	186	−247	−194	195	−130	131	−66	67		−31	246	9		22	−89	88
86	−185	184	−243	−196	197	−132	133	−68	69		−26	242	8		20	−87
−119	118	−183	182	−239	−198	199	−134	135	−70	71		−47	238	17		30
35	−117	116	−181	180	−235	−200	201	−136	137	−72	73		−51	234	16	
	46	−115	114	−179	178	−231	−202	203	−138	139	−74	75		−39	230	−7
15		19	−113	112	−177	176	−227	−204	205	−140	141	−76	77		−34	226
222	−6		29	−111	110	−175	174	−223	−206	207	−142	143	−78	79		−23
−50	252	14		36	−109	108	−173	172	−253	−208	209	−144	145	−80	81	
	−40	248	−5		45	−107	106	−171	170	−249	−210	211	−146	147	−82	83

Our Results

- We also show that there are at least $(n-2)[((k-11)/4)!/e]^2$ distinct such embeddings.


thank you