Weak L_1 inequalities for noncommutative singular integrals

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SINGULAR INTEGRALS OF CALDERÓN-ZYGMUND TYPE

- T is a Calderón-Zygmund operator on $\mathbb{R}^d$ if

$$Tf(x) \sim \int_{\mathbb{R}^d} K(x,y) f(y)\, dy$$

if T and K satisfy:

(i) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ (L^2-bound)

(ii) $|K(x,y)| \lesssim |x-y|^{-d}$ (size condition)

(iii) (Ideally) (additional smoothness)

$$\sup_{y,y'} \int_{\mathbb{R}^d \setminus B(y, 2|y-y'|)} |K(x,y) - K(x,y')|\, dx < \infty.$$

- T extends to a bounded operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$.
- Reason: $T : L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$

THE TECHNIQUE THAT PROVES $T: L^1 \to L^{1,\infty}$

Theorem (Calderón–Zygmund): Let $\lambda > 0$, $0 \leq f \in L^1(\mathbb{R}^d)$.
There exists a disjoint family $\{Q_j\}_j \subset \mathcal{D}$ such that:

(i) $\lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x)\,dx \leq 2^d \lambda$.

(ii) $|f(x)| \leq \lambda$ a.e $x \in \left(\bigcup_j Q_j \right)^c$

Corollary: $f = g + b$, so that:

(i) $|g(x)| \leq \lambda$ a.e $x \in \mathbb{R}^d$, $\|g\|_1 \lesssim \|f\|_1$.

(ii) $b = \sum_j b_j$, where $\mathrm{supp}(b_j) \subset Q_j$, $\int_{Q_j} b_j = 0$,
and $\sum_j \|b_j\|_1 \lesssim \|f\|_1$.

MATRIX VALUED FUNCTIONS

- Given a CZ Kernel K, we want to extend its action to functions $f : \mathbb{R}^d \to M_n(\mathbb{C})$. If $f = (f_{ij})_{i,j}$,

$$Tf(x) := (Tf_{ij}(x))_{i,j}$$

- Goal: weak (1,1) estimates for T with constants independent of n (so that we may extend to infinite matrices).

Rmk: Vector valued techniques cannot help. The constant must depend on n (related: L^1, L^∞ are not UMD).

SEMICOMMUTATIVE VON NEUMANN ALGEBRAS

- Idea: interpret $f : \mathbb{R}^d \to M_n(\mathbb{C})$ as elements of an algebra:

$$L^\infty(\mathbb{R}^d; M_n(\mathbb{C})) = L^\infty(\mathbb{R}^d) \overline{\otimes} M_n(\mathbb{C}) =: \mathcal{A}.$$

$$f \in \mathcal{A} \Rightarrow \varphi(f) := \int_{\mathbb{R}^d} \mathrm{Tr}((f_{ij})(x))\, dx.$$

Target weak L^1 inequality:

$$\sup_{\lambda > 0} \lambda \int_{\mathbb{R}^d} \underbrace{\mathrm{Tr}(\{|Tf| > \lambda\})}\, dx \lesssim \underbrace{\int_{\mathbb{R}^d} |\mathrm{Tr}(f)|\, dx}$$

Needs to be made sense of.

$\|f\|_{L_1(\mathcal{A})}$

$L_{1,\infty}(\mathcal{A}, \varphi)$

- If $f \in \mathcal{A}_+$, $f = \int_0^\infty \lambda \, de_f(\lambda)$.
- $\chi_{[\lambda,\infty)}(f) = \int_\lambda^\infty de_f(\lambda)$ (it represents $\{x : f(x) \geq \lambda\}$)
- $\|f\|_{L_{1,\infty}(\mathcal{A})} = \sup_{\lambda > 0} \lambda \cdot \varphi(\chi_{[\lambda,\infty)}(f))$.
- $L_{1,\infty}(\mathcal{A}) = \{ f \in \mathcal{A} : \|f\|_{L_{1,\infty}(\mathcal{A})} < \infty \}$
- Interpolation: $[L_{p_0}(\mathcal{A}), L_{1,\infty}(\mathcal{A})]_\theta = L_{p_\theta}(\mathcal{A})$

(noncommutative interpolation scale, not vector valued one).

CALDERÓN-ZYGMUND CUBES ON $\mathcal{A}$.

Thm (Cuculescu '71): Let $(\mathcal{M}, \tau)$ be a n.c. measure space, $\{\mathcal{M}_k\}$ a filtration of $\mathcal{M}$. Let $\lambda > 0$ and $0 \le x \in L_1(\mathcal{M})$. There exist projections $q_1, q_2, \ldots, q$ such that:

(i) $q_k E_k(x) q_k \le \lambda q_k$ and $q x q \le \lambda$ $\quad (q \sim \{x \le \lambda\})$

(ii) $q \le q_k \le q_{k-1}$ for all k. $\quad$ (q_k are nested)

(iii) $\tau(1_{\mathcal{M}} - q) \le \frac{1}{\lambda} \|x\|_{L_1(\mathcal{M})}$ $\quad$ (weak type for maximal function)

CUCULESCU ON $\mathcal{A}$, $\{\mathcal{A}_k\}$

- q_k is the projection that represents the set where $E_j(f) \leq \lambda$, for all $j \leq k$.

- Set $p_k := q_{k-1} - q_k$.

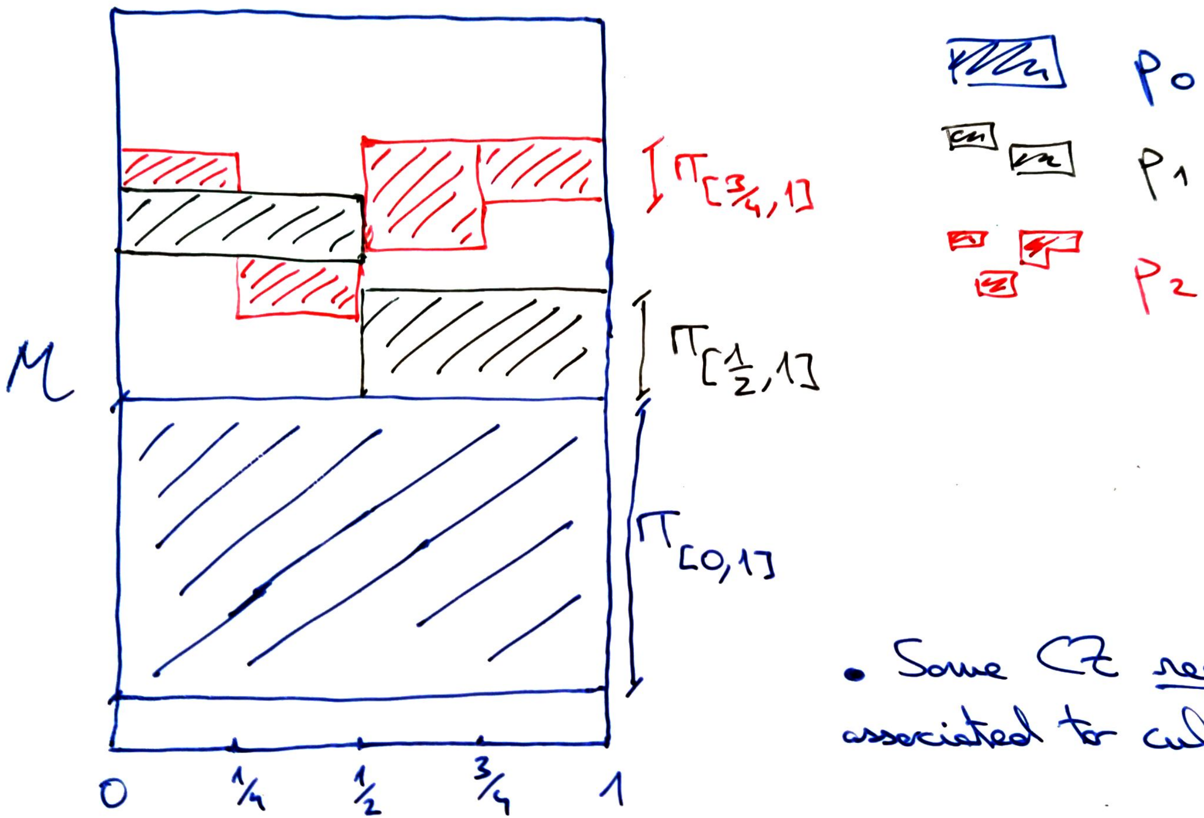
- p_k represents the dyadic cubes of side length 2^{-k} such that $E_k(f) > \lambda$ but $E_j(f) \leq \lambda$, $j \leq k$, i.e.

p_k represents the CZ cubes of side length 2^{-k}!

- <u>Remark</u>: on $\mathcal{A}$, $p_k = \sum_{Q \in \mathcal{D}_k} \pi_Q \chi_Q$, where $\pi_Q \in \mathcal{P}(\mathcal{M})$

$$\Longrightarrow$$

all cubes are CZ cubes on some part of $\mathcal{M}$!!



- Some CZ regions associated to cubes.

- Once we have CZ cubes one can try and build a CZ decomposition (previous works. Parcet '08, Cadilhac '18).

Thm (Cadilhac, C-A, Parcet): Let $0 \leq f \in L_1(\mathcal{A})$, and $\lambda > 0$. Let q_k, p_k be the Cuculescu projections of f at height λ. Then, we can split $f = g + b + f_{off}$, where:

(i) $g = q f q + \sum_k p_k E_k(f) p_k$, and $\|g\|_{L_\infty(\mathcal{A})} \lesssim \lambda$.

(ii) $b = \sum_k p_k (f - E_k(f)) p_k =: \sum_k b_k$, and $\sum_k \|b_k\|_1 \leq 2\|f\|_1$

(iii) $f_{off} = \sum_k (p_k f q_k + q_k f p_k) = \sum_k f_{off,k}$,

and $E_k(f_{off,k}) = 0$ . . . .

- Let K be a CZ Kernel satisfying

$$\sup_{Q\in\mathcal{D}} \sum_{j=1}^{\infty} \sup_{y\in Q} \left(2^{jd} \ell(Q)^d \int_{2^j \ell(Q) \leq |x-x_Q| \leq 2^{j+1}\ell(Q)} |K(x,y)-K(x,x_Q)|^2 \, dx \right)^{1/2} < \infty \quad (*)$$

<u>Theorem</u> (Cadilhac, C-A, Parcet): If K satisfies $(*)$,

$$T: L_1(\mathcal{A}) \to L_{1,\infty}(\mathcal{A}),$$

that is,

$$\sup_{\lambda>0} \lambda\, \varphi\left(\chi_{[\lambda,\infty)}(|Tf|)\right) \lesssim \varphi(|f|) \text{ for all } f\in L_1(\mathcal{A})$$

<u>Remark</u>: The theorem holds when $\mathcal{A} = L^\infty(\mathbb{R}^d)\,\overline{\otimes}\, \mathcal{M}$ for any von Neumann algebra $\mathcal{M}$ with a trace.

COROLLARIES AND REMARKS

- Smoothness condition (*) seems optimal for matrix valued functions, but we do not know.
- There is a version of the CZ decomposition for nonregular martingales (i.e. nondoubling measures):

$$f = qfq + \sum_k \left[p_k f p_k - E_{k-1}(p_k f p_k) \right] + \sum_k E_{k-1}(p_k f p_k)$$
$$+ \sum_k \left[p_k f q_k + q_k f p_k \right]$$

- Our results yield weak L_1 estimates for (radial) Fourier multipliers on l.c. groups (by transference).

THANK YOU VERY MUCH!

NAJLEPŠA HVALA!