On the linearity of order-isomorphisms

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The problem

A **cone** C in a vector space V induces a partial ordering $\leq$, where

$$x \leq y \quad \text{if} \quad y - x \in C.$$

Here C is convex, $\lambda C \subseteq C$ for all $\lambda \geq 0$, and $C \cap (-C) = \{0\}$.

$X \subseteq V$ and $Y \subseteq W$, where V and W are partially ordered vector spaces.

$f : X \to Y$ is an **order-isomorphisms**, if f is a bijection and

$$x \leq y \quad \text{if and only if} \quad f(x) \leq f(y).$$

So f and f^{-1} are both order-preserving.

Question When are such maps affine?

History

The question goes back to the 1950's and was motivated by special relativity.

Alexandrov and Ovčinnikova (1953): order-isomorphisms $f : C \to C$, where C is the finite dimensional Lorentz cone are linear.

Alexandrov (1967) Extension to more general finite dimensional cones.

Related works by Zeeman (1964), Rothaus (1966), Arstein-Avidan and Slomka (2012).

Noll and Schäffer extended Alexandrov's work to infinite dimensions in a sequence of papers (1977-1979).

Not always linear!

The map

$$f : x \mapsto x^3$$

is a **nonlinear** order-isomorphism on $\mathbb{R}$.

If (V, C) is a partially ordered vector space, then $W = V \oplus \mathbb{R}$ with cone $C \times \mathbb{R}_{\geq 0}$ admits a nonlinear order-isomorphism.

Can replace $\mathbb{R}_{\geq 0}$ by the cone of nonnegative functions in $C(K)$.

Question Which partially ordered vector spaces (V, C) admit a **nonlinear** order-isomorphism $f : C \to C$?

Recent progress on this question by Walsh for order-unit spaces

The domain

A subset X of V is called an **upper set** if for each $x \in X$ and $v \in C$ we have that $x + v \in X$.

Examples: V, C, or $\mathrm{int}\, C$.

We **only** consider order-isomorphisms $f : X \to Y$, where $X \subseteq V$ and $Y \subseteq W$ are **upper sets**.

Things are **different** (more complicated) when X or Y is not an upper set!

X and Y order-intervals: Drnovšek, Molnar, Mori, Šemrl, Roelands and Wortel

Some useful concepts

A cone C in V is **Archimedean** if for each $x \in V$ and $y \in C$ we have that

$$nx \leq y \text{ for all } n \geq 1 \implies x \leq 0.$$

An element $u \in C$ is called an **order-unit** if for each $x \in V$ there exists $\lambda \geq 0$ such that $x \leq \lambda u$.

$U \subseteq V$ is said to be **directed** if for each $x, y \in U$ there exists $z \in U$ with $x \leq z$ and $y \leq z$.

(V, C) is **directed** if V is a directed set, which is equivalent to $V = C - C$.

Notation

$$[x, z] = \{y \in V : x \leq y \leq z\} \quad \text{(order-interval)}$$

$$[x, \infty) = x + C = \{x + v : v \in C\} \quad \text{(cone with apex } x\text{)}.$$

Extreme rays

Given $x \in C$ with $x \neq 0$ we let

$$R_x = \{\lambda x \colon \lambda \geq 0\} \quad \text{(ray through } x\text{)}$$

R_x is said to be an **extreme ray** if $0 \leq y \leq x$ implies $y = \lambda x$ for some $0 \leq \lambda \leq 1$.

We call $x + R$ an **extreme half-line** if R is an extreme ray of C.

Order theoretic characterisation

Intuition An order-isomorphism should map extreme half-lines onto extreme half-lines.

Fact $H = x + R$ is an extreme half-line if and only if H is maximal among all subsets G of $[x, \infty)$ with $x \in G$ satisfying

(1) G is directed.

(2) For each $y \in G$, $[x, y]$ is totally ordered.

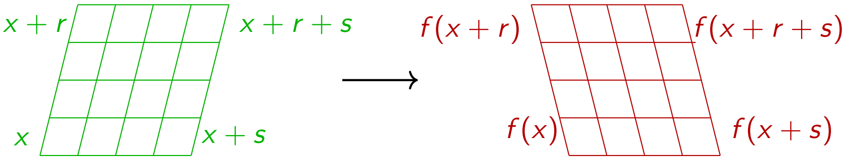
(3) G contains at least 2 distinct points.

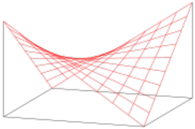
Corollary An order-isomorphism $f\colon X \to Y$ maps an extreme half-line $x + R$ onto an extreme half-line $f(x) + S$.

Key observation

Lemma Let R and S be distinct extreme rays of C and $f: X \to Y$ be an order-isomorphism. For $x \in X$, $r \in R$ and $s \in S$ we have that

$$f(x+r+s) - f(x+s) = f(x+r) - f(x).$$





Engaged extreme rays

Let $\mathcal{R}$ be the collection of all extreme rays of C.

Definition An extreme ray R of C is said to be **engaged** if

$$R \subseteq \mathrm{Span}\,(\mathcal{R} \setminus \{R\}) = \mathrm{Span}\{s\colon s \in S \text{ and } S \in \mathcal{R} \setminus \{R\}\}.$$

Otherwise, we say it is **disengaged**.

Let $\mathcal{R}_E$ be the collection of all engaged extreme rays.

In the Lorentz cone C with $\dim C \geq 3$, all rays in the boundary of C are extreme rays and engaged.

For $\mathbb{R}^n_+$ the extreme rays are $R_i = \{\lambda e_i \colon \lambda \geq 0\}$, which are all disengaged.

A consequence

Define

$$[x,\infty)_{\mathcal{R}_E} = \{x+r_1+\cdots+r_k \in [x,\infty) \colon r_i \in R_i \cup (-R_i),\ R_i \text{ engaged}\}.$$

Theorem If $f\colon [x,\infty) \to [y,\infty)$ is an order-isomorphism, then f is affine on $[x,\infty)_{\mathcal{R}_E}$.

Noll and Schäffer

Theorem Suppose $X \subseteq V$ and $Y \subseteq W$ are upper sets in Archimedean partially ordered vector spaces, where (V, C) is directed and $f: X \to Y$ is an order-isomorphism. If

$$C = \{r_1 + \cdots + r_k \colon r_i \in R_i \text{ extreme ray for all } i\}$$

and each $R \in \mathcal{R}$ is engaged, then f is affine.

Condition, $C = \{r_1 + \cdots + r_k \colon r_i \in R_i \text{ extreme ray for all } i\}$ holds if $\dim V < \infty$, but rarely in infinite dimensions.

Cones with too few extreme rays

The cone

$$C([0,1])_+ = \{f \in C([0,1])\colon f \geq 0\}$$

has **no** extreme rays.

The cone $B(H)^+_{\mathrm{sa}}$ of positive semidefinite operators has extreme rays $R_P = \{\lambda P\colon \lambda \geq 0\}$, where P is a rank-one projection.

If $\dim H = \infty$, then $B(H)^+_{\mathrm{sa}} \neq \mathrm{Span}_+\{R_P\colon P \text{ rank-one projection}\}$

inf's and sup's

Idea Order-isomorphisms preserve the inf's and sup's if they exist.

For $U \subseteq [a, \infty)$ and $x \in [a, \infty)$ we say that

$$x = \inf U \quad \text{in } [a, \infty),$$

if $x \leq u$ for all $u \in U$, and if $z \in [a, \infty)$ is such that $z \leq u$ for all $u \in U$, then $z \leq x$.

Similarly, $x = \sup U$ in $[a, \infty)$ is x is the least upper bound of U in $[a, \infty)$.

inf-sup-hull

Given $U \subseteq [a, \infty)$ the **inf-sup-hull** of U in $[a, \infty)$ is the set

$$\{x \in [a, \infty) \colon x = \inf_{\alpha \in A} \sup_{\beta \in B} u_{\alpha\beta} \text{ in } [a, \infty), \text{ where } u_{\alpha\beta} \in U\}.$$

Theorem (L., van Gaans, van Imhoff) Suppose $X \subseteq V$ and $Y \subseteq W$ are upper sets in Archimedean partially ordered vector spaces, where (V, C) is directed and $f \colon X \to Y$ is an order-isomorphism. If C equals the inf-sup-hull of $[0, \infty)_{\mathcal{R}_E}$, then f is affine.

Bounded self-adjoint operators

Theorem (Molnar) If H is a Hilbert space with $\dim H \geq 2$ and $X, Y \subseteq B(H)_{\mathrm{sa}}$ are upper sets, then every order-isomorphism $f\colon X \to Y$ is affine.

Easy to show that $B(H)_{\mathrm{sa}}^{+}$ is the inf-sup-hull (sup-hull) of the engaged extreme rays.

Corollary (Molnar) There is no order-isomorphism from $B(H)_{\mathrm{sa}}$ onto $\mathrm{int}\, B(H)_{\mathrm{sa}}^{+}$.

Extension to atomic JBW-algebras without a type I_1 part, Roelands and van Imhoff (2020).

Positive homogeneity

Question Is there a natural condition so that **all** order-isomorphisms are linear?

$f : \mathrm{int}\, C \to \mathrm{int}\, K$ is **positively homogeneous** if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathrm{int}\, C$ and $\lambda > 0$.

Theorem (Schäffer) If (V, C) and (W, K) are order-unit spaces, then every positively homogeneous order-isomorphism $f : \mathrm{int}\, C \to \mathrm{int}\, K$ is linear.

Spaces without order-unit

Theorem (L., van Gaans, van Imhoff) Suppose (V, C) and (W, K) are Archimedean partially ordered vector spaces, where (V, C) is directed and C is equal to the inf-sup-hull of

$$\{r_1 + \cdots + r_k \in C \colon r_i \in R_i \cup (-R_i) \text{ where } R_i \text{ is an extreme ray}\}.$$

Then every positive homogeneous order-isomorphism $f \colon C \to K$ is linear.

Results applies to $\ell_p(\mathbb{N})$ spaces (these spaces have no order unit).

Another related result

Theorem (L., van Gaans, van Imhoff) Let (V, C, u) and (W, K, e) be order unit spaces and $X \subseteq V$ and $Y \subseteq W$ be upper sets. Suppose that the inf-sup-hull of $[0,\infty)_{\mathcal{R}_E}$ has a non-empty intersection with $\mathrm{int}\, C$ and that either $(V, \|\cdot\|_u)$ or $(W, \|\cdot\|_e)$ is separable and complete. Then every order-isomorphism $f\colon X \to Y$ is affine.

An example

Let $V = C([0,1] \cup [2,3]) \oplus \mathbb{R}$ with cone

$$C = \{(f, \lambda) \colon \|f\|_\infty \leq \lambda\}.$$

Then $(V, C, (0,1))$ is a complete separable order-unit space.

The four points

$$(\pm \mathbb{1}_{[0,1]}, 1) \quad \text{and} \quad (\pm \mathbb{1}_{[2,3]}, 1)$$

correspond to the extreme rays, which are all engaged, since

$$(\mathbb{1}_{[0,1]}, 1) + (-\mathbb{1}_{[0,1]}, 1) = 2(0,1) = (\mathbb{1}_{[2,3]}, 1) + (-\mathbb{1}_{[2,3]}, 1).$$

. As $(0,1) \in \operatorname{int} C$, the theorem applies.

Thank you for your attention

B. Lemmens. O. van Gaans and H. van Imhoff, On the linearity of order-isomorphisms. *Canad. J. Math.* **73**(2), (2021), 399–416.