The Witten Conjecture for homology $S^1 \times S^3$

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1950s (Yang and Mills) Non-abelian version of the classical electromagnetic theory

1980s (Donaldson) Applications to topology of smooth 4-manifolds via the Donaldson polynomials

1988 (Witten) Quantum Field Theory: Donaldson polynomials = expectation values of certain observables. Degree-zero Donaldson polynomial = partition function.

1994 (Witten) The Seiberg–Witten invariants and the Witten Conjecture

Natural domain: closed oriented smooth 4-manifolds X with $\pi_1(X) = 1$ and $b_2^+(X) > 1$ (Feehan and Leness)

Extension: smooth 4-manifolds X with $b_2^+(X) = 1$ via wall-crossing formulas

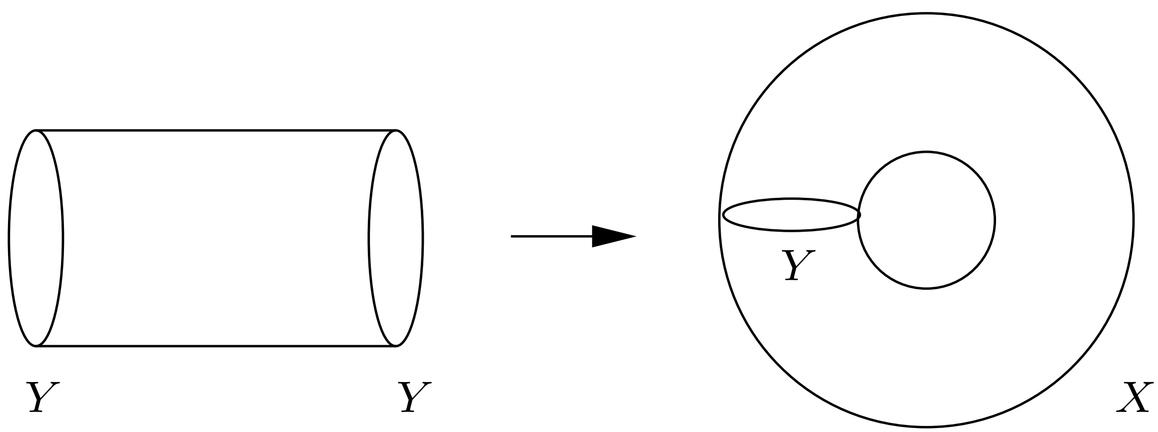
This project: smooth 4-manifolds X with $b_1(X) = 1$ and $b_2(X) = 0$

Homology $S^1 \times S^3$ is a smooth oriented closed spin 4-manifold X such that

$$H_*(X;\mathbb{Z}) = H_*(S^1 \times S^3;\mathbb{Z}).$$

Example. A product $X = S^1 \times Y$, where Y is an integral homology sphere.

Example. A "furled up" homology cobordism from Y to itself:



Homology orientation of X is a choice of generator $1 \in H^1(X;\mathbb{Z})$.

Donaldson Theory

Given a metric on X, consider connections A in the trivial $SU(2)$ bundle on X such that

$$F_A + \star F_A = 0.$$

These are called **instantons**. The moduli space $\mathcal{M}(X,g)$ of irreducible instantons is compact, oriented, and 0-dimensional (perhaps after perturbation).

Furuta–Ohta invariant

$$\lambda_{\mathsf{FO}}(X) = \frac{1}{4}\,\#\mathcal{M}(X,g).$$

Theorem. This is a well-defined diffeomorphism invariant of X if $H_*(\tilde{X};\mathbb{Q}) = H_*(S^3;\mathbb{Q})$, where $\tilde{X} \to X$ is the universal abelian cover.

The instantons in question are **flat** so $\lambda_{\mathsf{FO}}(X)$ can be viewed as a count of irreducible representations $\pi_1(X) \to \mathrm{SU}(2)$.

Product case: $\lambda_{\mathsf{FO}}(S^1 \times Y) = \lambda(Y)$.

Seiberg–Witten Theory

Given a metric g on X and a form $\beta \in \Omega^1(X, i\mathbb{R})$, consider the triples

$$(A, s, \varphi) \in \Omega^1(X, i\mathbb{R}) \times \mathbb{R}_{\geq 0} \times \mathbb{C}^\infty(S^+)$$

such that

$$\begin{cases} F_A^+ - s^2 \cdot \tau(\varphi, \varphi) = d^+\beta \\ D_A^+(X, g)(\varphi) = 0, \quad \|\varphi\|_{L^2(X)} = 1 \end{cases}$$

Seiberg–Witten moduli space $\mathcal{M}(X, g, \beta):$ the gauge equivalence classes of solutions of the above system. The solutions with $s = 0$ are called **reducible**.

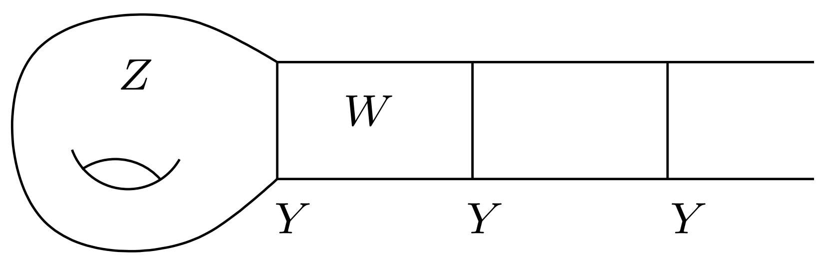
Theorem. For generic (g, β), the moduli space $\mathcal{M}(X, g, \beta)$ is a compact oriented 0-dimensional manifold with no reducibles.

Denote by $\#\,\mathcal{M}(X, g, \beta)$ the signed count of points in this moduli space. It depends on g and β.

Correction Term

Let $Y \subset X$ be a Poincaré dual to $1 \in H^1(X;\mathbb{Z})$ and cut X open along Y to obtain a cobordism W from Y to itself.

End-periodic manifold is a smooth manifold $Z_+ = Z \cup W \cup W \cup \ldots$ where Z is a compact smooth spin 4-manifold with $\partial Z = Y$



Product case: $X = S^1 \times Y$ gives rise to $Z_+ = Z \cup ([0,+\infty) \times Y)$. The index theory was studied by Atiyah, Patodi and Singer.

General case: the basics of index theory on Z_+ were established by Taubes. We developed this theory far enough to prove the following two theorems.

Theorem (Mrowka, Ruberman, S, 2011) For generic (g, β), the operator

$$D^+(Z_+, g) + \beta : L^2_1(Z_+) \to L^2(Z_+)$$

is Fredholm, and

$$w(X, g, \beta) = \text{ind}(D^+(Z_+, g) + \beta) + \text{sign}(Z)/8$$

is independent of the choice of Z and the way g and β are extended over $Z \subset Z_+$.

Theorem (Mrowka, Ruberman, S, 2011)

$$\lambda_{\text{SW}}(X) = \#\mathcal{M}(X, g, \beta) - w(X, g, \beta)$$

is a diffeomorphism invariant of X.

Product case: Weimin Chen (1997) and Yuhan Lim (1999).

Witten Conjecture

Conjecture (Mrowka, Ruberman, S, 2011) Let X be a homology $S^1 \times S^3$ such that $H_*(\tilde{X};\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. Then

$$\lambda_{\mathsf{FO}}(X) = -\lambda_{\mathsf{SW}}(X).$$

Theorem (Yuhan Lim, 1999) Let Y be an integral homology sphere. Then

$$\lambda_{\mathsf{FO}}(S^1 \times Y) = -\lambda_{\mathsf{SW}}(S^1 \times Y).$$

Theorem (Jianfeng Lin, Ruberman, S, 2020) Let Y be an integral homology sphere and X the mapping torus of a diffeomorphism $\tau : Y \to Y$ generating a semi-free action of $\mathbb{Z}/n$. Then

$$\lambda_{\mathsf{FO}}(X) = -\lambda_{\mathsf{SW}}(X).$$

The proof uses, on the Furuta–Ohta side, the equivariant Casson invariant of Collin–S (1999) and Ruberman-S (2004)

Explicit formulas

If τ has fixed points: $Y' = Y/\tau$ is an integral homology sphere and the quotient map $Y \to Y'$ is the n–fold branched cover with branch set a knot K. Then $\lambda_{\mathsf{FO}}(X) = -\lambda_{\mathsf{SW}}(X)$ equals

$$n \cdot \lambda(Y') + \frac{1}{8} \sum \mathsf{sign}^{m/n}(K)$$

(also proved by Langte Ma (2020) using a surgery formula for $\lambda_{\mathsf{SW}}(X)$).

If τ has no fixed points: $Y' = Y/\tau$ is a homology lens space which can be obtained by (n/q)–surgery along a knot K in an integral homology sphere Σ. Then $\lambda_{\mathsf{FO}}(X) = -\lambda_{\mathsf{SW}}(X)$ equals

$$n \cdot \lambda(\Sigma) + \frac{1}{8} \sum \mathsf{sign}^{m/n}(K) + \frac{q}{2}\,\Delta''_K(1).$$

Floer-theoretic interpretation

Assume that the Poincaré dual to $1 \in H^1(X;\mathbb{Z})$ can be chosen to be a rational homology sphere $Y \subset X$. Cut X open along Y to obtain W.

Theorem (Jianfeng Lin, Ruberman, S, 2018)

$$-\lambda_{\mathsf{SW}}(X) = h_{\mathsf{SW}}(Y) + \mathsf{Lef}\,(W_*),$$

where $h_{\mathsf{SW}}(Y)$ is the Frøyshov invariant and $\mathsf{Lef}(W_*)$ is the Lefschetz number in the reduced monopole Floer homology $HM^{\mathsf{red}}(Y)$ of Kronheimer and Mrowka.

Theorem (Anvari, 2019) If $Y \subset X$ is an integral homology sphere,

$$\lambda_{\mathsf{FO}}(X) = h_D(Y) + \mathsf{Lef}\,(W_*),$$

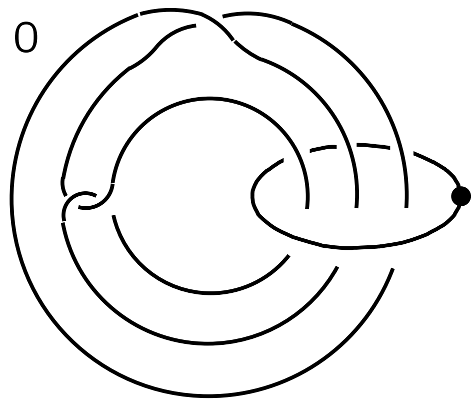
where $h_D(Y)$ is the instanton Frøyshov invariant and $\mathsf{Lef}(W_*)$ is the Lefschetz number in the reduced instanton Floer homology $I^{\mathsf{red}}(Y)$ of Frøyshov.

Topological Applications

Stem from calculating Lef(W_*) for mapping cylinders of self-diffeomorphisms

$$\tau : Y \to Y.$$

Example. Akbulut cork W_0



Smooth compact contractible 4-manifold with boundary Y. The boundary admits involution $\tau : Y \to Y$ exchanging the two link components.

Theorem (Akbulut, 1991) The involution τ does not extend to a diffeomorphism of W_0.

Theorem (Jianfeng Lin, Ruberman, S, 2018) The involution $\tau : Y \to Y$ does not extend to a diffeomorphism of **any** compact smooth 4-manifold W with boundary Y such that

$$H_*(W; \mathbb{Z}/2) = H_*(D^4; \mathbb{Z}/2).$$

Extended by Dai, Hedden, and Mallick (2020) to other involutions using the Floer-theoretic framework developed by Hendricks, Manolescu, and Zemke.

Theorem (Jianfeng Lin, Ruberman, S, 2018) Let $K \subset S^3$ be a Khovanov-thin knot and Y its double branched cover. Then

$$h(Y) = \frac{1}{8}\,\text{sign}(K).$$

This is known as the Manolescu–Owens Conjecture. It was proved by Manolescu–Owens for all alternating knots and by Lisca for all quasi-alternating knots.

Why $\lambda_{SW}(X)$ is metric independent

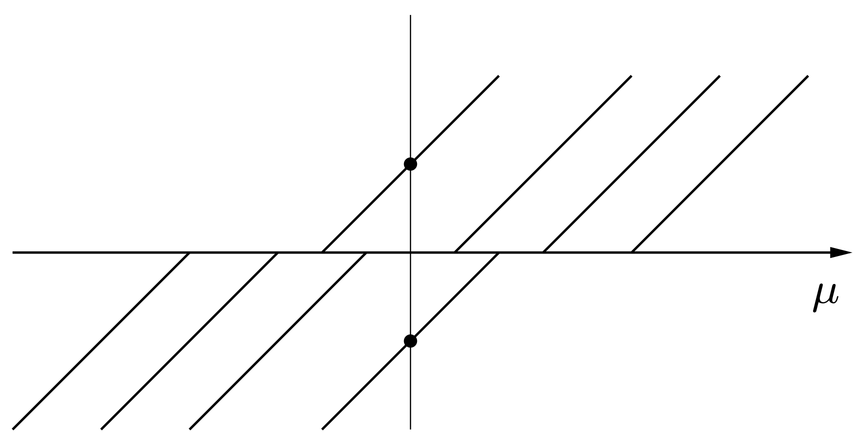
For simplicity, consider the product case $X = S^1 \times Y$ only. Then

$$\lambda_{\mathsf{SW}}(X) = \#\mathcal{M}(Y) + \frac{1}{2}\eta(Y) + \frac{1}{8}\eta_{\mathsf{Sign}}(Y).$$

For any $\mu \in \mathbb{R}$ consider the moduli space $\mathcal{M}(Y,\mu)$ of triples (A,φ) such that

$$F_A = \tau(\varphi,\varphi), \quad D_A(\varphi) = \mu\varphi.$$

A schematic depiction of $\mathcal{M}(Y) = \bigcup_\mu \mathcal{M}(Y,\mu)$



For any $t > 0$ a direct calculation shows that

$$\#\mathcal{M}(Y,0) + \frac{1}{2}\eta(Y) \;=\; \lim_{t\to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \#\mathcal{M}^*(Y,\mu) \cdot t^{1/2} \cdot e^{-t\mu^2}\, d\mu.$$

In particular,

$$\#\mathcal{M}(Y) + \frac{1}{2}\eta(Y)$$

is a continuous function of the metric. But $\eta_{\mathsf{Sign}}(Y)$ is also a continuous function of the metric. Since the sum

$$\#\mathcal{M}(Y) + \frac{1}{2}\eta(Y) + \frac{1}{8}\eta_{\mathsf{Sign}}(Y)$$

is an integer, it must be constant as a function of the metric.