The quintic NLS equation on the tadpole graph

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Nonlinear Schrödinger equation

Why nonlinear Schrödinger equation on graphs? And where?

- NLS: the equation

$$i\frac{\partial}{\partial t}\psi(x,t) = -\Delta\psi(x,t) + W(x)\psi(x,t) + \gamma|\psi(x,t)|^{2p}\psi(x,t)$$

where

- $x \in \mathbb{R}^n$, $t > 0$ and W is an external potential, possibly present;
- power nonlinearity $|\psi|^{2p}\psi$, the most common is $p = 1$ (the cubic case);
- $\gamma > 0$ defocussing, $\gamma < 0$ focusing (often in the following $\gamma = \pm(p+1)$)
- NLS: paradigm of nonlinear wave propagation: dispersion, scattering, bound states, breathers, solitons, stability of these discrete structures...;
- NLS: many physical systems described by NLS: Langmuir waves in plasma physics, e.m. pulse propagation in Kerr media, dynamics of BEC (Gross-Pitaevskii equation);
- NLS is Hamiltonian; when $n = 1$ and $p = 1$ and $W = 0$ also integrable
- NLS on graphs: Y-junctions, H-junctions or more complex structures. Some of them realized in BEC's. More complicated modellization in fiber optics arrays, where a more realistic description is however in terms of systems of NLS-type equations.
  See also N 2014 for a general overview of the subject (cited references are at the end of the slides)

Nonlinear Schrödinger Equation on graphs

$$i\frac{d}{dt}\Psi = H\Psi \pm (p+1)|\Psi|^{2p}\Psi$$

Linear term: H is a linear operator with δ-interaction in the vertices plus a potential

$$\mathcal{D}(H) := \left\{ \Psi \in H^2(\mathcal{G}) | \sum_{e \prec v} \partial\psi_e(v) = \alpha(v)\psi_e(v),\ \alpha(v) \in \mathbb{R},\ \forall v \in V \right\}.$$

$$H\Psi = -\Psi'' + W\Psi$$

and W fairly general (Cacciapuoti, Finco, N 2017)

Componentwise: $i\frac{d}{dt}\psi_e = -\frac{d^2}{dx_e^2}\psi_e + W_e\psi_e \pm (p+1)|\psi_e|^{2p}\psi_e \qquad \forall e \in E$ + B.C.

- $|E|$ scalar equations
- Coupled by the conditions in the vertices

Included in the above B.C. are the Neumann-Kirchhoff or natural B.C.: $\alpha(v) = 0$ or

$$\sum_{e \prec v} \partial\psi_e(v) = 0$$

With N-K boundary conditions we will write $H = -\Delta$.

Well posedness

A few words about the time dependent equation
(see Cacciapuoti Finco N 17 for more details)

With mild hypotheses on potentials and the above boundary conditions:

local well posedness of the strong solutions
(solutions with values in the operator domain $\mathcal{D}(H)$)

local well posedness of weak solutions
(solutions with values in the form domain $H^1(\mathcal{G})$)

Moreover for weak solutions the mass or L^2- norm,

$$M[\Psi] := ||\Psi||^2$$

is conserved, as well the energy

$$E[\Psi] = \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{\underline{v}\in V} \alpha(\underline{v})|\Psi(\underline{v})|^2 - \|\Psi\|_{2p+2}^{2p+2}$$

Global well posedness

- $p < 2$
- $p = 2$ for small masses (the critical case)
- in the special example of star graphs strong instability and blow-up of supercritical equation ($p > 2$) has been recently studied (Goloshchapova-Ohta, 2020)

Ground states

We call ground state a minimizer Φ of the energy E with fixed mass M.

$$E[\Phi] = \inf\{E[\Psi] \text{ s.t. } \Psi \in H^1(\mathcal{G}),\ M[\Psi] = \mu\} := \mathcal{E}_\mu \tag{1}$$

Notice that this definition contains two requirements:

- $\inf\{E[\Psi] \text{ s.t. } \Psi \in \mathcal{E},\ M[\Psi] = \mu\} > -\infty$
- The infimum is actually attained at some $\Phi \in H^1(\mathcal{G})$

Comments

- The existence of the ground state is usually considered as a good stability property of a physical system, significantly stronger than the mere boundedness from below of the energy. In particular ground states are orbitally stable.
- A minimizer does not necessarily exist, it does not necessarily exist for every mass, and when existing dependence from the mass could be relevant.
- Existence and properties of ground states in the case $W = 0$ and Neumann-Kirchhoff B.C. has been studied extensively and in depth by Adami-Serra-Tilli (2014-2017)
- Bifurcation of ground states from the bottom of the spectrum of the linear Hamiltonian has been studied in the generic case in Cacciapuoti-Finco-N 17

Euler-Lagrange equations

Any ground state Φ, as a constrained minimum point, satisfies, for some $\lambda \in \mathbb{R}$ (the Lagrange multiplier)

- $$-\phi_e'' - |\phi_e|^{2p}\phi_e + W_e(x)\phi_e = \lambda\phi_e \qquad \forall \text{ edge } e \qquad \text{(NLS)}$$

- $$\sum_{e \succ \underline{v}} \frac{d\phi_e}{dx_e}(\underline{v}) = \alpha(\underline{v})\phi_e(\underline{v}) \qquad \forall \underline{v} \qquad (\delta \text{ b.c.})$$

Actually the same set of equations and B.C. rules any constrained critical point, not only constrained minima. We call bound states constrained critical points.

Any bound state corresponds to a solution $\Psi(x,t)$ to the time dependent NLS s.t.

$$\Psi(x,t) \;=\; e^{-i\omega t}\Phi(x)$$

where ω takes the role of the Lagrange multiplier λ, and the profile Φ satisfies the stationary equation (written in compact form)

$$-\Delta\Phi + W\Phi - |\Phi|^{2p}\Phi = \omega\Phi \qquad \text{(sNLS)}$$

From now on we will be interested in the case $W = 0$ and Neumann Kirchhoff B.C.

Ground state and standing waves: the line

For $p \in (0,2)$ and $M = m > 0$ *ground states exist* and all of them are obtained translating a soliton. The soliton on the line is explicit:

$$\varphi_\omega(x) = |\omega|^{\frac{1}{2p}} \operatorname{sech}^{\frac{1}{p}}(p\sqrt{|\omega|}\, x) \qquad \omega < 0$$

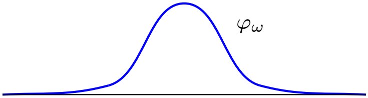


Figure: The soliton φ_ω.

In the case of the halfline $\mathcal{G} = \mathbb{R}^+$, $p \in (0,2)$ and every $\mu > 0$, there is *one and only one* ground state given by "half a soliton".

Notice that in this case the translational symmetry is broken by the Neumann b.c.

For the critical case $p = 2$ we have solitons of the same form as above, but now the mass is independent on ω

A concrete example: the tadpole graph

The tadpole graph

The tadpole graph $\mathcal{T}$ is the metric graph $\mathcal{G}$ constituted by a circle and a half-line attached at a single vertex.

We normalize the interval for the circle to $[-\pi, \pi]$ with the end points connected to the half-line $[0, \infty)$ at a single vertex.

The natural Neumann–Kirchhoff boundary conditions for the two-component vectors $\Phi := (u, v) \in H^2(-\pi, \pi) \times H^2(0, \infty)$ are given by

$$(BC) \quad \begin{cases} u(\pi) = u(-\pi) = v(0), \\ u'(\pi) - u'(-\pi) = v'(0). \end{cases}$$

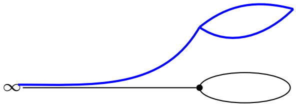
The Laplace operator $\Delta : \mathcal{D}(\Delta) \subset L^2(\mathcal{T}) \mapsto L^2(\mathcal{T})$ with the operator domain

$$\mathcal{D}(\Delta) := \left\{ \Phi = (u, v) |\ u \in H^2(-\pi, \pi), \ \ v \in H^2(0, \infty) : \ \ \text{satisfying } (BC) \right\} \qquad (2)$$

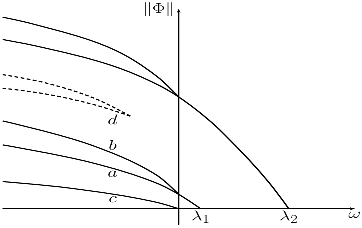
is self-adjoint in $L^2(\mathcal{T}) := L^2(-\pi, \pi) \times L^2(0, \infty)$.

The tadpole graph with subcritical power

- It is known that the subcritical ($0 < p < 2$) NLS equation for the tadpole graph $\mathcal{T}$ admits a ground state Φ *for all positive values of the mass* μ.
- The ground state Φ is given by a monotone piece of soliton on the half-line glued with a piece of a periodic (elliptic in the cubic or quintic case) function on the circle, with a single maximum at the antipodal point to the vertex



In the subcritical case (Cacciapuoti, Finco N '15, N Pelinovsky, Shaikova '15) a complex bifurcation diagram (more work needed for a complete understanding)



The tadpole graph with critical power (N-Pelinovsky '20)

In the following we will be interested in the NLS on the tadpole graph in the absence of potentials and with the critical power $p = 2$ (Noja-Pelinovsky '20)

- For $p = 2$ the ground state on any metric graph $\mathcal{G}$ with exactly one half-line (e.g., on the tadpole graph $\mathcal{T}$) is attained if and only if (Adami Serra Tilli '17)

$$\mu \in (m_{\mathbb{R}+}, m_{\mathbb{R}}]$$

- Soliton of the quintic NLS equation on the line centered at $x = 0$

$$\varphi_\omega(x) = |\omega|^{1/4}\text{sech}^{1/2}(2\sqrt{|\omega|}x)$$

$$m_{\mathbb{R}+} = \|\varphi_\omega\|^2_{L^2(\mathbb{R}+)} = \frac{\pi}{4}\,, \qquad m_{\mathbb{R}} = \|\varphi_\omega\|^2_{L^2(\mathbb{R})} = \frac{\pi}{2}.$$

- Notice that both values are independent on ω for $p = 2$
- So, the ground state on the tadpole graph $\mathcal{T}$ exists if and only if

$$\mu \in (m_{\mathbb{R}+}, m_{\mathbb{R}}] = (\frac{\pi}{4}, \frac{\pi}{2}] \qquad (\text{and} \;\; \mathcal{E}_\mu < 0)$$

- What happens above this range of masses (where $\mathcal{E}_\mu = -\infty$)?

The tadpole graph with critical power

Minimizing energy at constant mass is not the only variational problem giving information about standing waves.

An alternative constrained minimization problem is

$$\mathcal{B}(\omega) = \inf_{\Phi \in H^1(\mathcal{T})} \left\{ B_\omega(\Phi) : \quad \|\Phi\|_{L^6(\mathcal{T})} = 1 \right\}, \quad \omega < 0, \quad (\mathbf{S})$$

where

$$B_\omega(\Phi) := \|\nabla \Phi\|^2_{L^2(\mathcal{T})} - \omega \|\Phi\|^2_{L^2(\mathcal{T})}.$$

- The Euler Lagrange equations associated to this constrained variational problem is the stationary NLS equation (after scaling to adjust coefficients)

$$-\Delta \Phi - 3|\Phi|^4 u = \omega \Phi \qquad \text{(sNLS)}$$

- Versions of this variational problem arise in the determination of the best constant of the Sobolev inequality (equivalent to the Gagliardo–Nirenberg inequality in $\mathbb{R}^n$, but not on a metric graph)
- The above variational problem gives generally a larger set of standing waves compared to the set of ground states

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The tadpole graph with critical power

Theorem

For every $\omega < 0$*, there exists a global minimizer* $\Phi_\omega \in H^1(\mathcal{T})$ *of the constrained minimization problem* **S**. *By regularity, this yields a strong solution to the stationary NLS equation.*
Φ_ω *is real up to the phase rotation, positive up to sign choice, symmetric on* $[-\pi, \pi]$ *and monotonically decreasing on* $[0, \pi]$ *and* $[0, \infty)$.

Main steps of the proof

- $B_\omega(\Phi) := \|\nabla\Phi\|^2_{L^2(\mathcal{T})} - \omega\|\Phi\|^2_{L^2(\mathcal{T})}$ is equivalent to the $H^1(\mathcal{T})$ norm
- from $||\Phi||_6 = 1$ one has $\mathcal{B}(\omega) = \inf_{\Phi\in H^1(\mathcal{T})}\{B_\omega(\Phi)\} > 0$
- a minimizing sequence $\{\Phi_n\}$ satisfying $||\Phi_n||_6 = 1$ and $B_\omega(\Phi_n) \to \mathcal{B}(\omega)$ has a weak limit Φ_*. By Fatou Lemma, $0 \le ||\Phi_*||_6 \le \lim ||\Phi_n||_6 = 1$; let $\gamma := ||\Phi_*||_6$
- if $\gamma \in (0, 1)$ the sequence splits; ruled out.
- if $\gamma = 0$ the sequence vanishes: ruled out by a counterexample of a Φ_0 with $||\Phi_0||_6 = 1$ and $B_\omega(\Phi_0) < \min_{\Phi\in H^1(\mathbb{R})} B_\omega(\Phi, \mathbb{R})$ (proven not possible)
- it follows $\gamma = 1$, Φ_* is a strong limit of $\{\Phi_n\}$ and a minimizer
- restoring ω dependence one set $\Phi_* = \Phi_\omega$; regularity and B.C. are standard
- symmetry follows from Polya-Szegö inequality on metric graphs

The minimizer: mass-frequency relation

We want to understand the behavior of the family of standing waves Ψ_ω in terms of the mass, so giving relation with the problem of ground states.

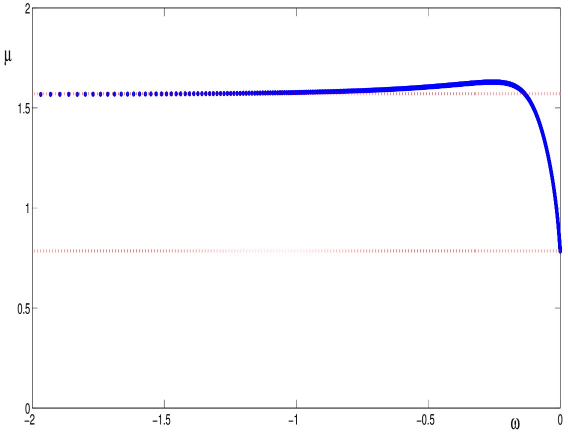

Figure: The horizontal dotted lines show the limiting levels $m_{\mathbb{R}+}$ and $m_{\mathbb{R}}$.

The minimizer: mass-frequency relation

- It is natural to introduce the following Lagrangian function ("action", more often) to treat our constrained variational problems

$$S_\omega(\Psi) = E(\Psi) - \omega M[\Psi]$$

- Notice that $S'_\omega \Psi = 0$ is nothing that the stationary equation, solved by Φ_ω

  To give information on local constrained stationary points (such that $S'_\omega \Psi_\omega = 0$) study second order variation of the action at the critical point Φ_ω

$$S''_\omega(\Phi_\omega)\eta = \langle L_1\alpha, \alpha\rangle + \langle L_2\beta, \beta\rangle$$

  where $\eta = \alpha + i\beta \cong (\alpha, \beta)$ and (for general p)

$$L_1 = -\Delta - \omega - (2p+1)(p+1)\Phi_\omega^{2p}$$
$$L_2 = -\Delta - \omega - (p+1)\Phi_\omega^{2p}$$

- If

$$(L_1\alpha, \alpha) \geq C_1||\alpha||^2_{H^1}, \qquad (L_2\beta, \beta) \geq C_2||\beta||^2_{H^1}$$

  one would have a local constrained minimum for the energy

  (the higher order remainder is easily under control)

The minimizer: mass-frequency relation

However things are more complicated.

- L_2 is annihilated by Φ_ω, because
$$L_2\Phi_\omega = (-\Delta - \omega - (p+1)\Phi_\omega^{2p})(\Phi_\omega) = 0$$
coincides with the stationary equation
- L_1 has at least a negative eigenvalue, because
$$\langle L_1\Phi_\omega, \Phi_\omega\rangle = -2p(p+1)||\Phi_\omega||_6^6 < 0$$
- what we really need is that S''_ω is positive on the *constrained* space
$$L_c^2 = \{u \in L^2| \;\; \langle u, \Phi_\omega\rangle = 0\}$$
- Now $\mathrm{Ker}(L_2) = \{\Phi_\omega\}$ and so L_2 is positive on L_c^2
- L_1 has a single negative eigenvalue, but the eigenvector is not Φ_ω
- However, if $\frac{d}{d\omega}||\Phi_\omega||^2 < 0$ (the Vakhitov-Kolokolov condition) one can show that the eigenvalue of L_1 on L_c^2 disappears
- Remaining issues:
  - determination of the kernel of L_1
  - monotonicity of mass $m(\omega) = ||\Phi_\omega||^2$ as a function of frequency ω

The minimizer: stability and mass-frequency relation

Analysis of the two issues

- Non degeneracy of L_1

Theorem

Let $\Phi_\omega \in \mathcal{D}(\Delta)$ be a solution to the stationary quintic NLS equation for $\omega < 0$ constructed by the previous variational problem. Then $Ker(L_1) = \emptyset$.
Moreover $\sigma_e(L_1) = [|\omega|, \infty)$.

The proof of the triviality of $\mathrm{Ker}(L_1)$ is rather direct and not immediate. Use of dynamical system techniques

- Monotonicity of $\omega \mapsto m(\omega)$

Theorem

Let $\Phi_\omega \in \mathcal{D}(\Delta)$ be the solution to the stationary NLS equation for $\omega < 0$ previously constructed.

Then, the mapping $\omega \mapsto m(\omega) = ||\Phi_\omega||^2$ is C^1 for every $\omega < 0$ and satisfies $m(\omega) \to m_{\mathbb{R}+}$ as $\omega \to 0$ and $m(\omega) \to m_{\mathbb{R}}$ as $\omega \to -\infty$.

Moreover, there exist a single ω_1 with $-\infty < \omega_1 < 0$ and $m(\omega_1) > m_{\mathbb{R}}$ such that $m'(\omega) > 0$ for $\omega \in (-\infty, \omega_1)$ and $m'(\omega) < 0$ for $\omega \in (\omega_1, 0)$.

The proof make heavy use of properties of modified Jacobi elliptic functions

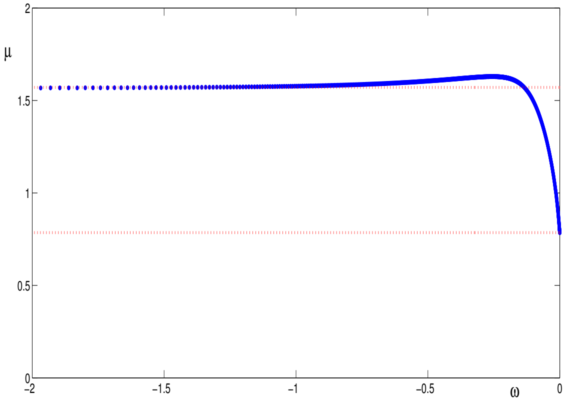

Figure: The horizontal dotted lines show the limiting levels $m_{\mathbb{R}+}$ and $m_{\mathbb{R}}$.

Theorem

The standing wave Φ_ω is a local minimizer of the energy $E(\Psi)$ subject to the constraint $M(\Psi) = \mu(\omega)$ for $\omega \in (\omega_1, 0)$; it is a saddle point of the energy $E(\Psi)$ subject to the constraint $M(\Psi) = \mu(\omega)$ for $\omega \in (-\infty, \omega_1)$.

Remarks and conclusions

- Ground states are orbitally stable for the NLS flow (Cazenave-Lions argument).
- Using the action as a Lyapunov function (see the classical theory of Weinstein '86 and Grillakis-Shatah-Strauss '87) one concludes that the standing waves $e^{-i\omega t}\Phi_\omega$ are orbitally stable for the NLS flow for $\omega \in (\omega_1, 0)$ (local constrained minima of the energy are orbitally stable)
- In particular, being $m(\omega_1) > m_{\mathbb{R}} := m(\omega_0)$, to the range $\omega \in (\omega_1, \omega_0)$ correspond stable standing waves that are not ground states
- Uniqueness of ground states is treated in Dovetta-Serra-Tilli 20
- A different situation, the line with a pendant, is treated in Pierotti-Soave-Verzini '2021
- Much to do as regards the complete scenario; many other bifurcations arise as the direct analysis of the stationary equation of the cubic NLS on the tadpole reveals
- Extension to flower graphs (several circles attached to the half-line end) due to Kairzhan-Marangell-Pelinovsky-Xiao (2021)

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Thanks for the attention!

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