Geometry of Kato manifolds

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-Topics in complex and quaternionic geometry-
Portorož, June 23rd, 2021

Plan of the talk:

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Interesting non-Kähler manifolds

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Theorem (Miyaoka, Todorov, Siu, Buchdahl, Lamari)

(M, J) compact complex surface admits a Kähler metric $\Leftrightarrow$ b_1 even.

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What about their higher dimensional analogues?

Kato manifolds = compact complex manifolds of $\dim_{\mathbb{C}} \geq 2$ admitting a global spherical shell

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$(V \simeq S_\epsilon := \{z \in \mathbb{C}^n \mid 1 - \epsilon < ||z|| < 1 + \epsilon\}, \epsilon > 0)$.

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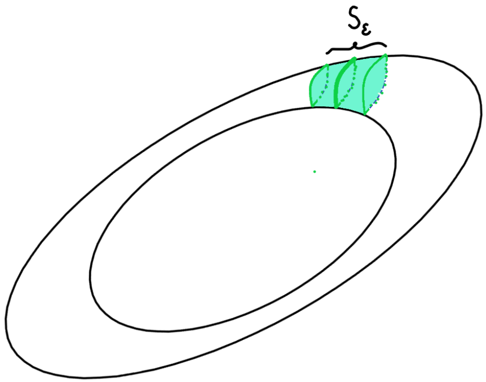
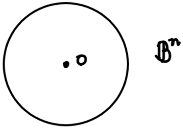
A *global spherical shell* (GSS) is a spherical shell such that $M \setminus V$ is connected.


S_ε

Kato manifolds



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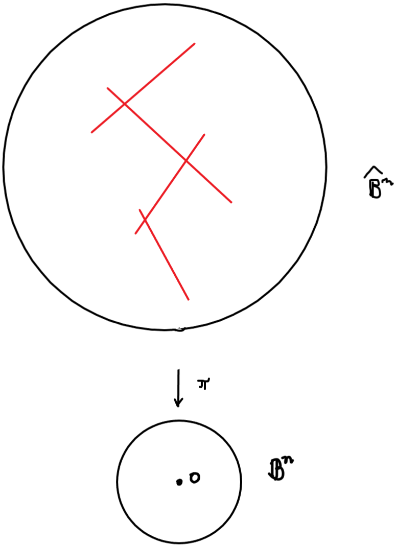

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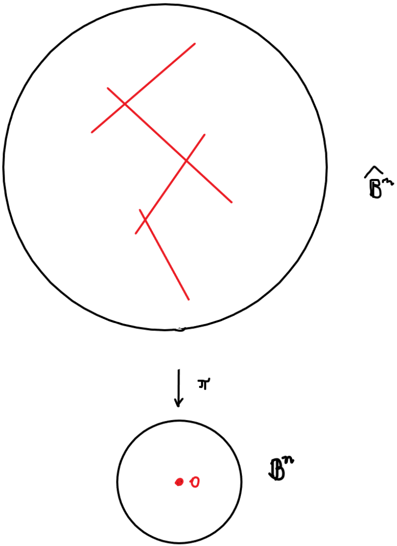

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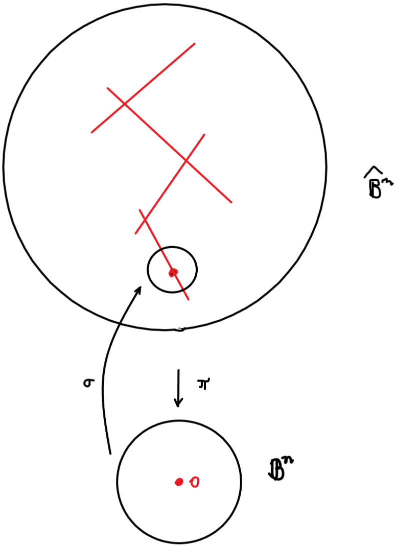

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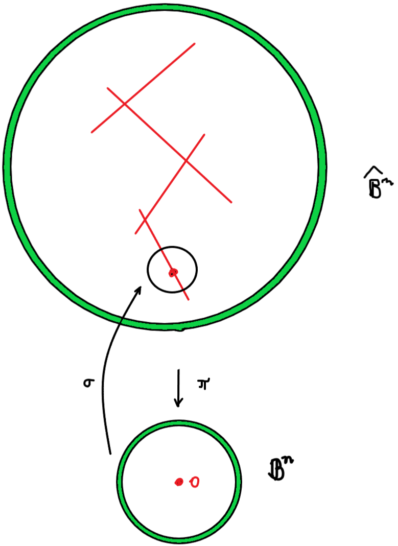

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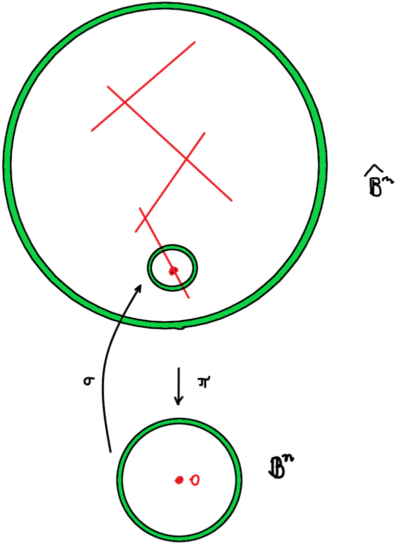

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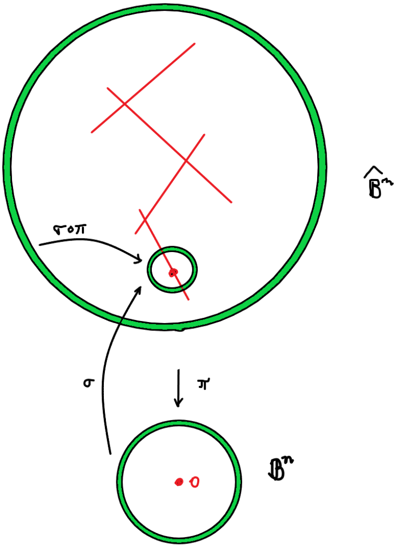

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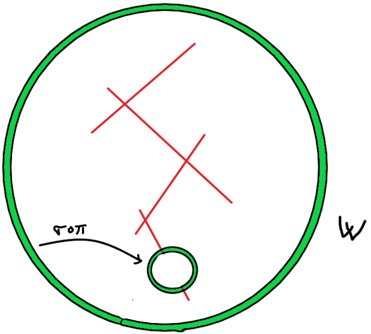

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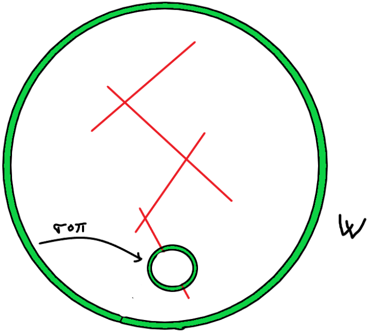

$X(\pi, \sigma) = W / \sigma \circ \pi$

Manifolds with global spherical shell

Theorem (Kato, '77)

Any compact manifold of $\dim_{\mathbb{C}} \geq 2$ *containing a GSS is obtained in the following way: Let* $\mathbb{B} \subset \mathbb{C}^n$, $\mathbb{B} := \{z \in \mathbb{C}^n \mid ||z|| < 1\}$

- *Let* $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ *be a modification at a finite number of points:*
- *Let* $\sigma : \overline{\mathbb{B}} \to \hat{\mathbb{B}}$ *be a holomorphic embedding.*
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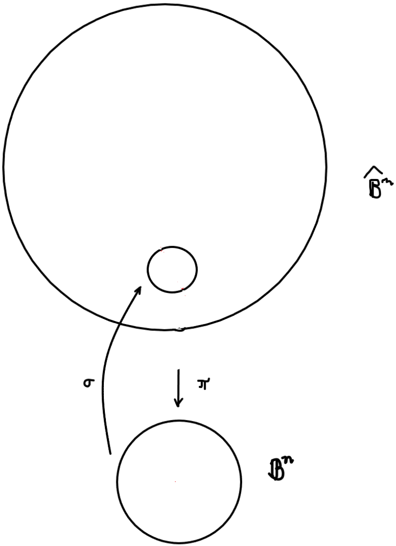
(π, σ) Kato data $\Rightarrow X(\pi, \sigma)$

Special case $\pi = \mathrm{id}$



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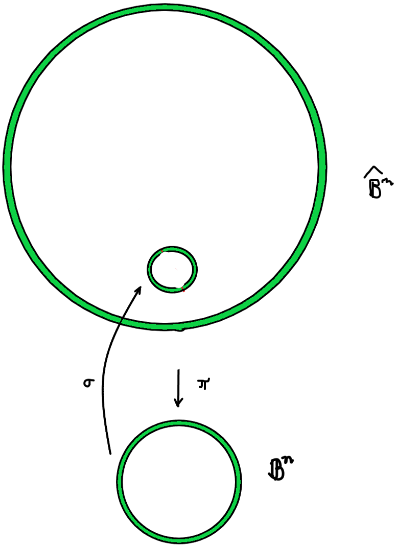

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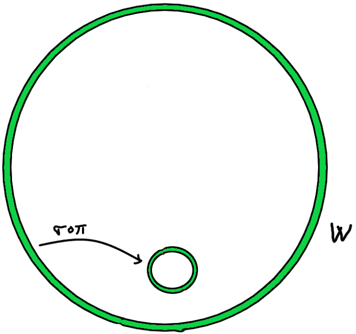

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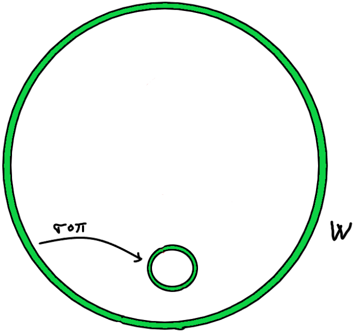

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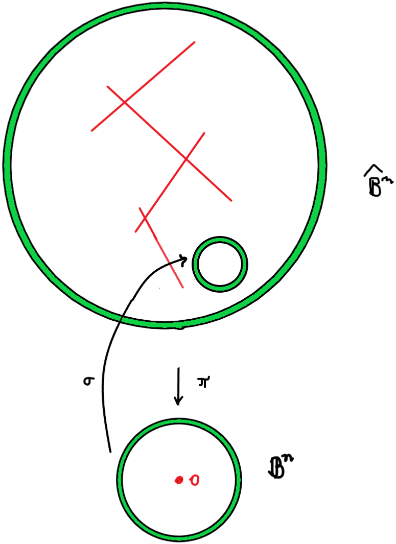

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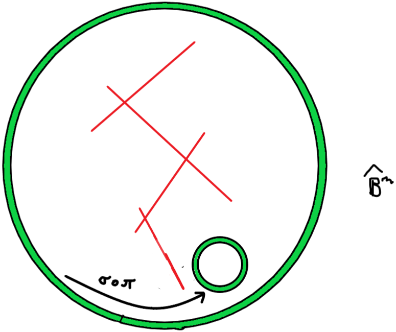

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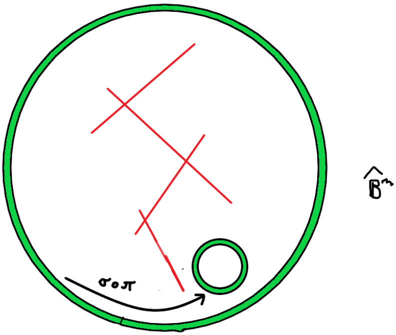
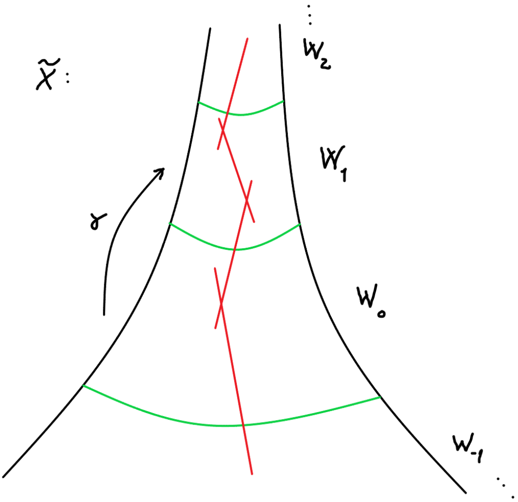


$X(\pi, \sigma)$ modification of a Hopf manifold

The universal cover:



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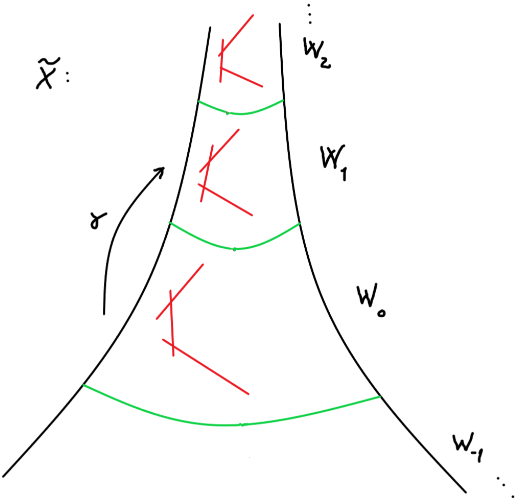

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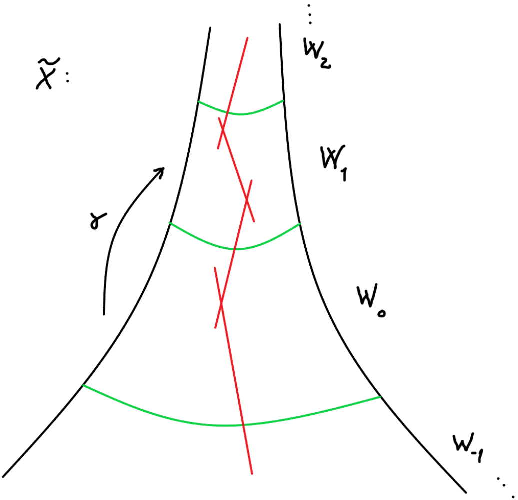

$\pi_1(M) \simeq \mathbb{Z}(\Rightarrow b_1 = 1)$ cannot support Kähler metrics.

Theorem (Kato, '77)

For any $X(\pi, \sigma)$, there exists a flat deformation $p : \mathcal{X} \to \mathbb{D}$ such that $p^{-1}(0) \simeq X(\pi, \sigma)$ and $p^{-1}(t)$ is a modification of a Hopf manifold.

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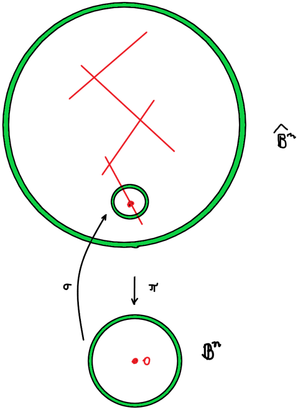

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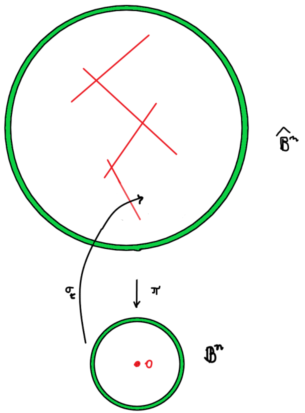

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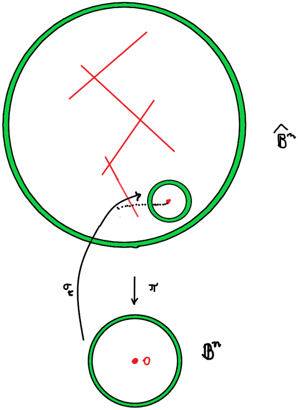

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5. Theorem (Brunella, '11): Any Kato surface admits locally conformally Kähler metrics.

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Ω is

- balanced if $d\Omega^{n-1} = 0$ (Michesohn)
- pluriclosed if $\partial\overline{\partial}\Omega = 0$ (Bismut)
- strongly Gauduchon if $\partial\Omega^{n-1}$ is $\overline{\partial}$-exact (Popovici)
- Hermitian symplectic if Ω is the $(1,1)$-part of a closed 2-form (Streets, Tian).

LCK Kato manifolds

Theorem (Istrati, -, Pontecorvo, Ruggiero, '20)

$X(\pi, \sigma)$ admits a locally conformally Kähler metric if and only if $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ is a Kähler modification.

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- example of non-Kähler $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ (in $\dim_{\mathbb{C}} \geq 3$) based on Hironaka's examples.

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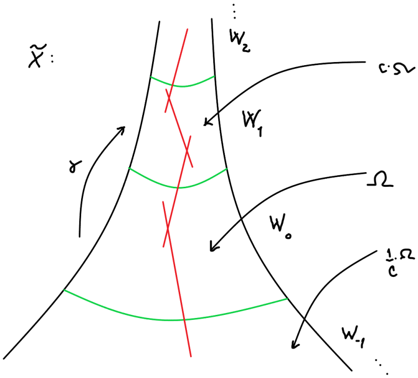

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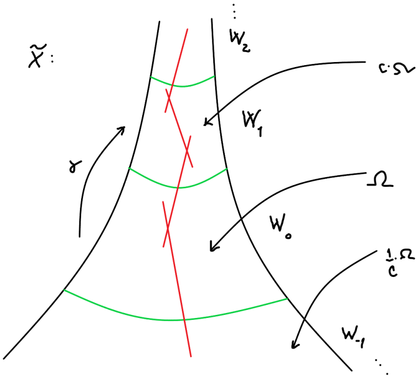

Conversely, show that $\hat{\mathbb{B}} \setminus \sigma(\{0\})$ is Kähler.

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- (Yau, Zhao, Zheng, 19): Hermitian symplectic $\Rightarrow$ strongly Gauduchon

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$\Rightarrow$ start to consider some special cases/impose some symmetries.

Special cases

- baby case (N. Istrati, -, M. Pontecorvo, '19): $X(\pi, \sigma)$ in the special case when $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ is a composition of blow-ups in points and σ is a blow-up chart.

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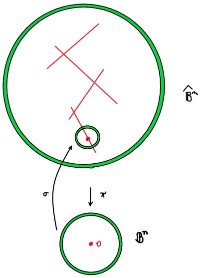

Key: give an equivalent construction!

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$$A \in \mathrm{GL}_n(\mathbb{Z}) \Rightarrow \begin{cases} \text{toric Kato manifolds of hyperbolic type} \\ \text{toric Kato manifolds of parabolic type} \\ \text{primary Hopf manifolds} \end{cases}$$

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2. hyperbolic case $\Rightarrow$ $h^{1,2} = 0 \Rightarrow$ no pluriclosed metrics in $\dim_{\mathbb{C}} \geq 3$.

Thank you very much for your attention!