Multiple oscillating BV-solutions for a mean-curvature Neumann problem

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The 1D mean curvature problem

High multiplicity result for the problem

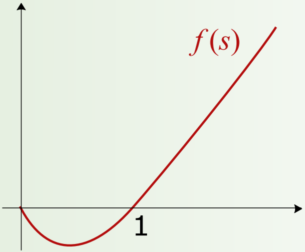
$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = w(x)f(u) & \text{in } (0,R) \\ u > 0 & \text{in } (0,R) \\ u'(0) = 0 = u'(R) \end{cases}$$

- $w \in C^1([0,R])$, $w > 0$
- $f \in C^1([0,\infty))$

(f_{eq}) $f(0) = 0 = f(1)$

(f_{sgn}) $f(s) < 0$ for $s \in (0,1)$
$f(s) > 0$ for $s > 1$

(f_{ap}) $\int_1^\infty f(s)ds = \infty$ or
$\|w\|_{L^1}|\min f| < 1$



prototype $f(s) = s^p - s$, $p > 1$

A reference problem

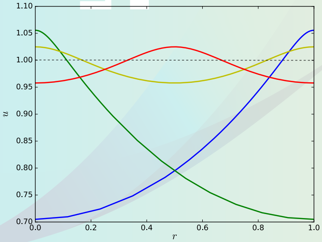
$$\begin{cases} -\Delta u = f(u) & \text{in } B_R \\ u > 0,\ u \text{ radial} & \text{in } B_R \\ \partial_\nu u = 0 & \text{on } \partial B_R \end{cases} \qquad f \in C^1([0,\infty))$$

Multiple oscillating C^2-solutions via shooting method

Theorem [A.Boscaggin, F.C., B.Noris, ESAIM '18 & PRSE '20]

Under (f_{eq}) and (f_{sgn}). If $f'(1) > \lambda_{k+1}^{\text{rad}}(R)$:

- $\exists\, k$ sol $u_1^-, \ldots, u_k^-$, $u_j^+ - 1$ has j zeros, $u_j^-(0) < 1$
- if f is <u>subcritical</u>, $\exists\, k$ sol $u_1^+, \ldots, u_k^+$, $u_j^+ - 1$ has j zeros, $u_j^+(0) > 1$



For the model case $f(s) = s^p - s$:

- Bonheure, Grumiau, Troestler, NA '16
- Bonheure, Grossi, Noris, Terracini, JDE '16

The 1D mean curvature problem

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \sim \begin{cases} \Delta u & \text{for } |\nabla u| \text{ small} \\ \Delta_1 u & \text{for } |\nabla u| \text{ large} \quad \rightsquigarrow \text{ work in } W^{1,1} \text{ ?} \end{cases}$$

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A different phenomenon appears already in dimension one

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = w(x)f(u) & \text{in } (0,R) \\ u > 0 & \text{in } (0,R) \\ u'(0) = 0 = u'(R) \end{cases}$$

even in the autonomous case $w(x) \equiv$ Const.

Solutions may present jump discontinuities at points where $|u'| = \infty$

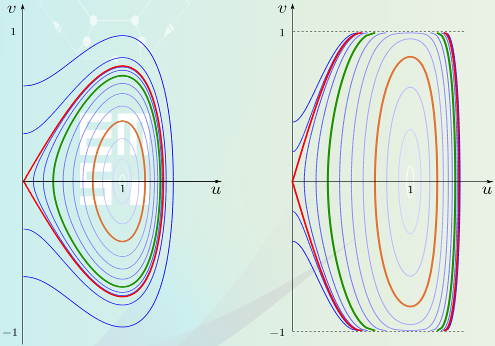
▷ Notice that $W^{1,1}(0,R) \subset C([0,R]) \rightsquigarrow$ we work in $BV(0,R)$

Autonomous case $(w(x) \equiv \bar{w})$

$\triangleright$ $f(s) = s^p - s \ \ (p > 1)$, $\quad -1 \le v := \frac{u'}{\sqrt{1+u'^2}} \le 1$, $\quad \varphi(s) := \frac{s}{\sqrt{1+s^2}}$

$$\begin{cases} v' = -\bar{w}\, f(u) \\ u' = \varphi^{-1}(v) \end{cases} \qquad u > 0, \quad v(0) = 0 = v(R)$$

Phase plane



* sol (u, v) turn around $(1, 0)$
* sol (u, v) parametrize energy level sets
* Neumann sol inside the homoclinic
* regular sol may coexsist with discontinuous ones

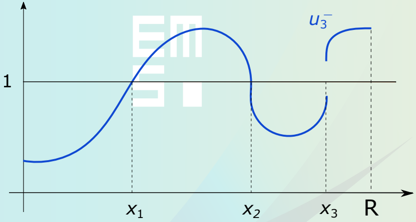
- Sufficient cond. for $|v| < 1$: $\bar{w}\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$

Theorem [A. Boscaggin, F. C., C. De Coster, JDE 2021]

Under the previous assumptions. If $f'(1) > \lambda_{k+1}$ (eigenvalue problem: $-u'' = \lambda w(x) u$ Neumann b.c.) then there exist

- k BV-solutions $u_1^-, \ldots, u_k^-$ with $u_j^-(0) < 1$
- k BV-solutions $u_1^+, \ldots, u_k^+$ with $u_j^+(0) > 1$

and $u_j^\pm - 1$ changes sign exactly j times in $(0, R)$.



[Lopez-Gomez, Omari, ANS '19]

* $u_j^\pm \in C^2(x_{i-1}, x_i)$
  x_i: "intersection points" of u_j with 1
* at jumps $|(u_j^\pm)'| = \infty$

- $N = 1 \Rightarrow -u''$ has the same sign of f

$$\left(\frac{r^{N-1}u'}{\sqrt{1+u'^2}}\right)' = \frac{r^{N-1}u''}{(1+u'^2)^{3/2}} + (N-1)\frac{r^{N-2}u'}{\sqrt{1+u'^2}}$$

at intersections u changes convexity

Definition of BV-solution

$$-(\varphi(u'))' = w(x)f(u), \quad \varphi(s) = \frac{s}{\sqrt{1+s^2}}$$

- $u \in W^{1,1}(0,R)$ weak solution if

$$\int_0^R \varphi(u')\psi' dx - \int_0^R w(x)f(u)\psi\, dx = 0 \quad \forall\, \psi \in C^\infty([0,R])$$

$\Leftrightarrow$ $u \in W^{1,1}(0,R)$ critical point of the C^1 convex functional

$$I_u(\psi) := \int_0^R \Phi(\psi')dx - \int_0^R w(x)f(u)\psi\, dx, \quad \Phi(s) = \sqrt{1+s^2} - 1$$

$\Leftrightarrow$ $u \in W^{1,1}(0,R)$ global minimizer of I_u: $\forall\, \psi \in C^\infty([0,R])$

$$\int_0^R \sqrt{1+u'^2}dx \le \int_0^R \sqrt{1+\psi'^2}dx - \int_0^R w(x)f(u)(\psi - u)dx$$

$u \in BV(0,R)$ is a BV-solution if for every $\psi \in C^\infty([0,R])$

$$\int_0^R \sqrt{1+|Du|^2} \le \int_0^R \sqrt{1+\psi'^2}dx - \int_0^R w(x)f(u)(\psi - u)dx,$$

$\int_0^R \sqrt{1+|Du|^2} = \int_0^R \sqrt{1+|Du^a|^2}dx + \int_0^R |Du^s|$

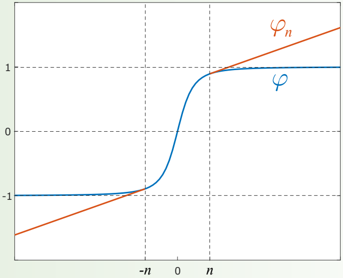
Idea of the proof

❶ Approximating problems

$$(P_n)\begin{cases}-(\varphi_n(u'))' = w(x)f(u)\\ u'(0) = 0 = u'(R)\end{cases}$$

$$\left(\varphi(u') = \frac{u'}{\sqrt{1+u'^2}} \le 1\right)$$

Multiplicity via shooting



❷ A priori estimates: (u_n) sol's of (P_n) with the same oscillatory behavior $\Rightarrow$ (u_n) bdd in $W^{1,1}(0,R)$

❸ Passing to the limit: $\exists\, u \in BV(0,R)$ s.t. $u_n \to u$ in L^1
The limit u is a BV sol of the original problem

❹ u preserves the oscillatory behavior of u_n We improve the convergence result away from intersection pt.s

Further results

- u has continuous energy
$$E(x) = 1 - \frac{1}{\sqrt{1+u'^2}} + w(x)\int_1^u f(s)ds$$
(information not included in the definition of BV-solution)
- sufficient condition for classical solutions:
$$w(0)\exp\left(\int_0^R \frac{w'^{+}(x)}{w(x)}dx\right)\int_1^0 f(s)ds < 1$$

Open question

Existence of non-constant radial solutions in $B_R \subset \mathbb{R}^N$, $N > 1$?

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(u) & \text{in } B_R \\ u > 0, \quad u \text{ radial} & \text{in } B_R \\ \partial_\nu u = 0 & \text{on } \partial B_R \end{cases}$$

Thank you
for your attention!