# Quantum information theory and Reznick's Positivstellensatz

Ion Nechita (CNRS, LPT Toulouse)
— joint work with Alexander Müller-Hermes and David Reeb

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# Talk outline

Sums of squares and Reznick's Positivstellensatz

Polynomials vs. symmetric operators

The complex Positivstellensatz

# Sums of squares and Reznick's Positivstellensatz

# Hilbert's 17th problem

$\mathbb{R}[x] \ni P(x) \geq 0 \iff P = Q_1(x)^2 + Q_2(x)^2$, for $Q_{1,2} \in \mathbb{R}[x]$.

$\mathrm{Pos}(d,n) := \{P \in \mathbb{R}[x_1,\ldots,x_d]$ hom. of deg. $2n$, $P(x) \geq 0$, $\forall x\}$.

$\mathrm{SOS}(d,n) := \{\sum_i Q_i^2$ with $Q_i \in \mathbb{R}[x_1,\ldots,x_d]$ hom. of deg. $n\}$.

In general, $\mathrm{SOS}$ is a strict subset of $\mathrm{Pos}$ [Hil88]

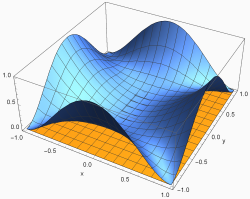
$$\mathrm{SOS}(d,n) \subseteq \mathrm{Pos}(d,n), \text{ eq. iff } (d,n) \in \{(d,1),(2,n),(3,2)\}.$$

The Motzkin polynomial $x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$ is positive but not SOS.

Membership in $\mathrm{SOS}$ can be efficiently decided with a semidefinite program (SDP) and provides an algebraic certificate for positivity.

# More on the Motzkin polynomial

The non-homogeneous Motzkin polynomial (set $z = 1$) $x^4y^2 + y^4 + x^2 - 3x^2y^2$ can be seen to be positive by the AMGM inequality.




There exist computer algebra packages to check SOS and perform polynomial optimization using SOS ([NC]SOSTOOLS, Gloptipoly)

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» syms x y z; findsos(x^4*y^2 + y^4 + x^2 - 3*x^2*y^2)

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...
No sum of squares decomposition is found.
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# Reznick's Positivstellensatz

Artin's solution to Hilbert's 17th problem [Art27]

$$P \geq 0 \iff P = \sum_i \frac{Q_i^2}{R_i^2}$$

In particular, if $P \geq 0$, there exists $R$ such that $R^2 P$ is SOS

**Theorem ([Rez95])**

*Let $P \in \mathrm{Pos}(d,k)$ such that $m(P) := \min_{\|x\|=1} P(x) > 0$. Let also $M(P) := \max_{\|x\|=1} P(x)$. Then, for all*

$$n \geq \frac{dk(2k-1)}{2\ln 2}\frac{M(P)}{m(P)} - \frac{d}{2},$$

*we have*

$$(x_1^2 + \cdots + x_d^2)^{n-k} P(x) = \sum_{j=1}^{r} (a_1^{(j)} x_1 + \cdots a_d^{(j)} x_d)^{2n}.$$

*In particular, $\|x\|^{2(n-k)} P$ is SOS.*

# Polynomials vs. symmetric operators

# From the symmetric subspace to polynomials — $\mathbb{R}$

Homogeneous polynomials of degree $n$ in $d$ real variables $x_1, \ldots, x_d$ are in one-to-one correspondence with symmetric tensors:

$$\vee^n \mathbb{R}^d \ni v \rightsquigarrow P_v(x_1, \ldots, x_d) = \langle x^{\otimes n}, v \rangle$$

where $x = (x_1, \ldots, x_d)$ is the vector of variables.

Examples:

- $n = 1,\ P_v(x) = \sum_{i=1}^d v_i x_i$;
- $|GHZ\rangle = |000\rangle + |111\rangle \rightsquigarrow P_{|GHZ\rangle}(x, y) = x^3 + y^3$;
- $|W\rangle = |001\rangle + |010\rangle + |001\rangle \rightsquigarrow P_{|W\rangle}(x, y) = 3x^2 y$;
- if $|\Omega\rangle = \sum_{i=1}^d |ii\rangle$, then $P_{|\Omega\rangle^{\otimes n}}(x_1, \ldots, x_d) = (\sum_{i=1}^d x_i^2)^n = \|x\|^{2n}$.

We denote $d[n] := \dim \vee^n \mathbb{R}^d = \binom{n+d-1}{n}$ [Har13].

# From the symmetric subspace to polynomials — $\mathbb{C}$

In the complex case, we are interested in bi-homogeneous polynomials of degree $n$ in $d$ complex variables: $P(z_1, \ldots, z_d)$ is hom. in the variables $z_i$ and also in $\bar{z}_i$.

Bi-hom. polynomials are in one-to-one correspondence with operators on $\vee^n \mathbb{C}^d$:

$$P(z_1, \ldots, z_d) = \langle z^{\otimes n} | W | z^{\otimes n} \rangle.$$

Self-adjoint $W$ are associated to real, bi-hom. polynomials.

The norm: $\|z\|^{2n} = \langle z^{\otimes n} | P_{sym}^{(d,n)} | z^{\otimes n} \rangle$.

More generally, polynomials which are bi-hom. of degree $n$ in complex variables $z_1, \ldots, z_d$ and, separately, bi-hom. of degree $k$ in complex variables $u_1, \ldots, u_D$ are in one-to-one correspondence with operators on $\vee^n \mathbb{C}^d \otimes \vee^k \mathbb{C}^D$:

$$Q(z_1, \ldots, z_d, u_1, \ldots, u_D) = \langle z^{\otimes n} \otimes u^{\otimes k} | W | z^{\otimes n} \otimes u^{\otimes k} \rangle.$$

# The different notions of positivity

A self-adjoint matrix $W \in \mathcal{B}(\vee^n \mathbb{C}^d)$ is called:

- block-positive if $\langle z^{\otimes n} | W | z^{\otimes n} \rangle \geq 0$, $\forall z \in \mathbb{C}^d$;
- positive semidefinite (PSD) if $\langle u | W | u \rangle \geq 0$, $\forall u \in \vee^n \mathbb{C}^d$;
- separable if $W \in \mathrm{conv}\{|z\rangle\langle z|^{\otimes n}\}_{z \in \mathbb{C}^d}$.

We have: $W$ separable $\Longrightarrow$ $W$ PSD $\Longrightarrow$ $W$ block-positive.

$W$ is block-positive $\iff$ $P_W$ is non-negative:

$$P_W(z) = \langle z^{\otimes n} | W | z^{\otimes n} \rangle \geq 0, \qquad \forall z \in \mathbb{C}^d.$$

$W$ is PSD $\iff$ $P_W$ is Sum Of hom. Squares:

$$W = \sum_j \lambda_j |w_j\rangle\langle w_j| \implies P_W(z) = \sum_j \lambda_j |\langle z^{\otimes n}, w_j \rangle|^2.$$

$W$ is separable $\iff$ $P_W$ is Sum Of hom. Powers:

$$W = \sum_j t_j |a_j\rangle\langle a_j|^{\otimes n} \implies P_W(z) = \sum_j t_j |\langle z, a_j \rangle|^{2n}.$$

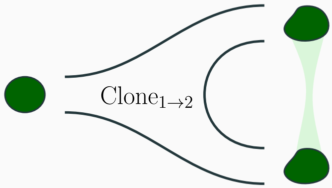
# Tensoring with the identity

For $k \leq n$, let $\mathrm{Tr}^*_{k\to n} : \mathcal{B}(\vee^k \mathbb{C}^d) \to \mathcal{B}(\vee^n \mathbb{C}^d)$ be the map

$$\mathrm{Tr}^*_{k\to n}(W) = P^{(d,n)}_{sym} \left[ W \otimes I_d^{\otimes(n-k)} \right] P^{(d,n)}_{sym}.$$

We have: $P_{\mathrm{Tr}^*_{k\to n}(W)}(z) = \|z\|^{2(n-k)} P_W(z)$.

$\mathrm{Clone}_{k\to n} := \frac{d[k]}{d[n]} \mathrm{Tr}^*_{k\to n}$ is the optimal Keyl-Werner cloning quantum channel [Wer98, KW99]: among all quantum channels sending states $\rho^{\otimes k}$ to symmetric $n$-partite states $\sigma$, it is the one which achieves the largest fidelity between $\rho$ and $\mathrm{Tr}_{2\dots n}\,\sigma$.



# The partial trace

For $k \leq n$, let $\mathrm{Tr}_{n\to k} : \mathcal{B}(\vee^n \mathbb{C}^d) \to \mathcal{B}(\vee^k \mathbb{C}^d)$ be the partial trace

$$\mathrm{Tr}_{n\to k}(W) = \left[\mathrm{id}^{\otimes k} \otimes \mathrm{Tr}^{\otimes (n-k)}\right](W).$$

**Lemma**

*We have:* $P_{\mathrm{Tr}_{n\to k}(W)} = ((n)_{n-k})^{-2} \Delta_{\mathbb{C}}^{n-k} P_W$, *where* $(x)_p = x(x-1)\cdots(x-p+1)$ *and* $\Delta_{\mathbb{C}}$ *is the complex Laplacian*

$$\Delta_{\mathbb{C}} = \sum_{i=1}^{d} \frac{\partial^2}{\partial \bar{z}_i \partial z_i}$$

**Lemma (complex Bernstein inequality ← we need analysis here)**

*For any* $W = W^* \in \mathcal{B}(\vee^n \mathbb{C}^d)$ *we have*

$$\forall \|z\| \leq 1, \qquad \left|(\Delta_{\mathbb{C}}^s P_W)(z)\right| \leq 4^{-s}(2d)^s (2n)_{2s} M(W)$$

| The Dictionary | |
|---|---|
| Sym. operators $\in \mathcal{B}(\vee^n \mathbb{C}^d)$ | Polynomials ($d$ vars, bi-hom. deg. $n$) |
| $W$ | $P_W(z) = \langle z^{\otimes n} \vert W \vert z^{\otimes n} \rangle$ |
| **Positivity notions** | |
| block-positive | non-negative |
| positive semidefinite | Sum Of Squares |
| separable | Sum Of Powers |
| **Operations** | |
| Tensor with identity | mult. with the norm$^2$ |
| Partial trace | complex Laplacian |

# The complex Positivstellensatz

# A complex version of Reznick's PSS

**Theorem ([MHNR19])**

*Consider $W = W^* \in \mathcal{B}(\vee^k \mathbb{C}^d \otimes \mathbb{C}^D)$ with $m(W) > 0$ and $k \geq 1$. Then, for any*

$$n \geq \frac{dk(2k-1)}{\ln\left(1 + \frac{m(W)}{M(W)}\right)} - k$$

*with $n \geq k$, we have*

$$\|x\|^{2(n-k)} P_W(x, y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x \rangle|^{2n} \mathrm{d}\varphi$$

*with $P_{\tilde{W}}(\varphi, y) \geq 0$ for all $\varphi \in \mathbb{C}^d$ and $y \in \mathbb{C}^D$, where the matrix $\tilde{W} \in \mathcal{B}(\vee^k \mathbb{C}^d \otimes \mathbb{C}^D)$ is explicitly computable in terms of $W$, and $\mathrm{d}\varphi$ is any $(n+k)$-spherical design. In the case $k = 1$, the bound on $n$ can be improved to $n \geq dM(W)/m(W) - 1$.*

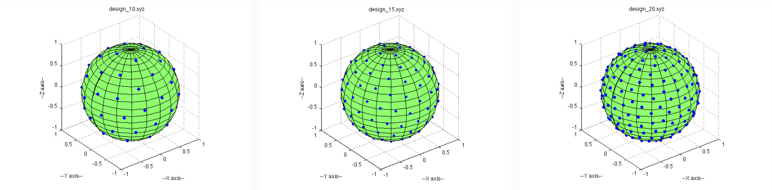
A similar result was obtained by To and Yeung [TY06] with worse bounds and in a less general setting, by "complexifying" Reznick's proof.

# Spherical designs

A complex $n$-spherical design in dimension $d$ [DGS91] is a probability measure $\mathrm{d}\varphi$ on the unit sphere of $\mathbb{C}^d$ which approximates the uniform measure $\mathrm{d}z$ in the following sense: for any degree $n$ bi-hom. polynomial $P(z)$ in $d$ complex variables, $\int P(\varphi)\mathrm{d}\varphi = \int P(z)\mathrm{d}z$. Equivalently,

$$\int |\varphi\rangle\langle\varphi|^{\otimes n}\mathrm{d}\varphi = \int_{\|z\|=1} |z\rangle\langle z|^{\otimes n}\mathrm{d}z = \frac{P_{sym}^{(d,n)}}{d[n]}.$$

For all $d, n$, there exist finite $n$-designs: the measure $\mathrm{d}\varphi$ has support of size $\leq (n+1)^{2d}$; in particular, the integral in the main theorem can be a finite sum




Designs of orders 60, 120, 216 in $\mathbb{R}^3$ ©John Burkardt

# Proof idea

$$\|x\|^{2(n-k)} P_W(x,y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x \rangle|^{2n} \mathrm{d}\varphi$$

- We want to transform a non-negative polynomial into a sum of powers by multiplying with some power of the norm.
- In terms of operators, this amounts to transforming a block-positive operator into a separable operator.
- Ansatz: use the measure-and-prepare map

$$\begin{aligned} \mathrm{MP}_{n\to k} : \mathcal{B}(\vee^n \mathbb{C}^d) &\to \mathcal{B}(\vee^k \mathbb{C}^d) \\ X &\mapsto d[n] \int \langle \varphi^{\otimes n} | X | \varphi^{\otimes n} \rangle |\varphi\rangle\langle\varphi|^{\otimes k} \mathrm{d}\varphi, \end{aligned}$$

  for some $(n+k)$-spherical design $\mathrm{d}\varphi$.
- The linear map $\mathrm{MP}_{n\to k}$ is completely positive, and it is normalized to be trace preserving (i.e. it is a quantum channel).

# Chiribella's identity

**Theorem ([Chi10])**

*For any $k \leq n$, we have*

$$\mathrm{MP}_{n\to k} = \sum_{s=0}^{k} c(n,k,s)\,\mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{n\to s},$$

*where* $c(n,k,s) = \binom{n}{s}\binom{k+d-1}{k-s} / \binom{n+k+d-1}{k}$.

Above, $c(n,k,\cdot)$ is a probability distribution: $\sum_{s=0}^{k} c(n,k,s) = 1$.

The proof of the Chiribella identity is a straightforward computation in the group algebra of $G = \mathcal{S}_{n+k}$:

$$\varepsilon_G = \sum_{s=0}^{\min(n,k)} \frac{\binom{n}{s}\binom{k}{s}}{\binom{n+k}{n}} \varepsilon_H \sigma_s \varepsilon_H$$

where $\varepsilon_X$ is the average of the elements in $X$, $H = \mathcal{S}_n \times \mathcal{S}_k \leq G$ is a Young subgroup and $\sigma_s$ is some permutation swapping $s$ elements from $[1,n]$ with $s$ elements from $[n+1, n+k]$.

## The result is about the interplay between Clone and MP

The equality $\|x\|^{2(n-k)} P_W(x,y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x\rangle|^{2n} \mathrm{d}\varphi$ reads, in terms of linear maps over symmetric spaces

$$\mathsf{Clone}_{k\to n} \otimes \mathsf{id}_D = [\mathsf{MP}_{k\to n} \circ \Psi] \otimes \mathsf{id}_D \,.$$

The fact that the polynomial $P_{\tilde{W}}$ is non-negative reads

$$\tilde{W} := \Psi(W) \text{ is block-positive } \iff \langle z^{\otimes n} | \tilde{W} | z^{\otimes n} \rangle \geq 0.$$

Re-write the Chiribella identity as

$$\begin{aligned}
\mathsf{MP}_{n\to k} &= \sum_{s=0}^{k} c(n,k,s)\, \mathsf{Clone}_{s\to k} \circ \mathsf{Tr}_{n\to s} \\
&= \sum_{s=0}^{k} c(n,k,s)\, \mathsf{Clone}_{s\to k} \circ \mathsf{Tr}_{k\to s} \circ \mathsf{Tr}_{n\to k} \\
&= \Phi^{(n)}_{k\to k} \circ \mathsf{Tr}_{n\to k} \,.
\end{aligned}$$

# Invert the Chiribella formula

Recall that $\mathrm{MP}_{n\to k} = \Phi^{(n)}_{k\to k} \circ \mathrm{Tr}_{n\to k}$, for some linear map $\Phi^{(n)}_{k\to k}$.

**Key fact.**

The linear map $\Phi^{(n)}_{k\to k} : \vee^k \mathbb{C}^d \to \vee^k \mathbb{C}^d$ is invertible, with inverse

$$\Psi^{(n)}_{k\to k} := \sum_{s=0}^{k} q(n,k,s)\, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{k\to s}$$

with

$$q(n,k,s) := (-1)^{s+k} \frac{\binom{n+s}{s}\binom{k}{s}}{\binom{n}{k}} \frac{d[k]}{d[s]}$$

Hence, up to some constants, $\mathrm{Clone}_{k\to n} = \mathrm{MP}_{k\to n} \circ \Psi^{(n)}_{k\to k}$.

Final step: use hypotheses on $n, k, m(W), M(W)$ to ensure $\Psi^{(n)}_{k\to k}(W)$ is block-positive whenever $W$ is (strictly) block-positive.

# Use the Bernstein inequality to prove $P_{\tilde{W}}$ non-negative

Assume, wlog, $D = 1$, i.e. there is no $y$. We have

$$\begin{aligned} P_{\tilde{W}}(\varphi) &= \sum_{s=0}^{k} q(n,k,s)\langle \varphi^{\otimes k} | \operatorname{Clone}_{s\to k} \circ \operatorname{Tr}_{k\to s}(W) | \varphi^{\otimes k}\rangle \\ &= \sum_{s=0}^{k} q(n,k,s)\|\varphi\|^{2(k-s)}\langle \varphi^{\otimes s} | \operatorname{Tr}_{k\to s}(W) | \varphi^{\otimes s}\rangle \\ &= \sum_{s=0}^{k} q(n,k,s)\|\varphi\|^{2(k-s)} P_{\operatorname{Tr}_{k\to s}(W)}(\varphi) \\ &= \sum_{s=0}^{k} \hat{q}(n,k,s)\|\varphi\|^{2(k-s)} (\Delta_{\mathbb{C}}^{k-s} p_W)(\varphi). \end{aligned}$$

Use the complex version of the Bernstein inequality to ensure that

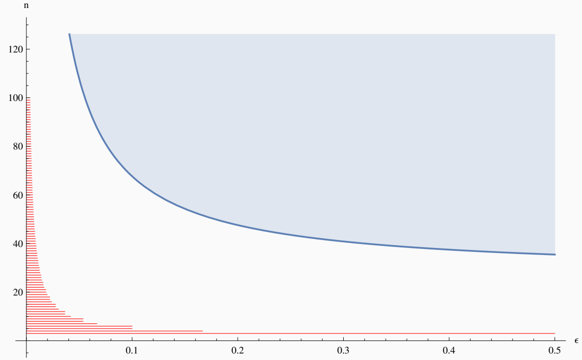
$$P_{\tilde{W}}(\varphi) \geq \left[ m(W)\tilde{q}(n,k,k) - M(W) \sum_{s=0}^{k-1} |\tilde{q}(n,k,s)| \right] \geq 0.$$

# How good are the bounds?

Consider the modified Motzkin polynomial

$$P_\varepsilon(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 + \varepsilon(x^2+y^2+z^2).$$

We have $m(P_\varepsilon) = \varepsilon$; $M(P_\varepsilon) = \varepsilon + 4/27$. Multiply with denominator $P_{n,\varepsilon}(x,y,z) := (x^2+y^2+z^2)^{n-3} P_\varepsilon(x,y,z)$. If a PSS decomposition for $P_{n,\varepsilon}$ exists, then the $[2p,2q,2r]$ coefficient of $P_{n,\varepsilon}$ must be positive $\rightsquigarrow$ lower bound on optimal $n$.



# The take-home slide

$W \in \mathcal{B}^{\mathrm{sa}}(\vee^n \mathbb{C}^d) \rightsquigarrow$ hom. poly. in $d$ vars of deg. $n$ $P_W(z) = \langle z^{\otimes n}|W|z^{\otimes n}\rangle$

$W$ is block-positive $\iff$ $P_W$ is non-negative.

$W$ is PSD $\iff$ $P_W$ is Sum Of hom. Squares:

$$W = \sum_j \lambda_j |w_j\rangle\langle w_j| \implies P_W(z) = \sum_j \lambda_j |\langle z^{\otimes n}, w_j\rangle|^2.$$

$W$ is separable $\iff$ $P_W$ is Sum Of hom. Powers:

$$W = \sum_j t_j |a_j\rangle\langle a_j|^{\otimes n} \implies P_W(z) = \sum_j t_j |\langle z, a_j\rangle|^{2n}.$$

**Theorem ([MHNR19])**

*For any* $W \in \mathcal{B}^{\mathrm{sa}}(\vee^k \mathbb{C}^d \otimes \mathbb{C}^D)$ *and* $n \geq [dk(2k-1)]/\ln\left(1 + \frac{m(W)}{M(W)}\right) - k$,

$$\|x\|^{2(n-k)} P_W(x,y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x\rangle|^{2n} \mathrm{d}\varphi \in \mathrm{SOP}(x) \subseteq \mathrm{SOS}(x),$$

*where the polynomial* $P_{\tilde{W}}(\cdot,\cdot) \geq 0$.

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