Observability for non-autonomous systems

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$\dot{x}(t) = \boxed{A(t)}x(t)\,,\ y(t) = C(t)x(t)\,,\ t>0\,,\quad x(0)=x_0\in X$

Example: $A(t)$ strongly ellip. Diff. op. on L^p

$C(t) = 1_{\Omega(t)}$ Mult. op.

Question: If we sample $y(t)$ at time $E \subseteq [0,T]$ can estimate $x(T)$

Final-state Obs.

$$\|x(T)\|_X \lesssim \int_E \|y(t)\|_Y \,dt$$

$A(t): \underline{D(A)} \to X$, X B-Space

$C(t) \in \mathcal{L}(X,Y)$, Y B-Space

"Solution operator" for the CP

Evo Fam. $U(t,0)$

$U(\cdot,0)x := x(\cdot) \in L^1([0,T],D) \cap W^{1,1}([0,T],X)$

Suff. Cond. for Obs.

- Exp. boundedness
- $(C(t))$ bounded fam.

$$\|U(t,0)x\| \le \|(I-P_\lambda)U(t,0)x\| + \|P_\lambda U(t,0)x\|$$

$\lesssim e^{-\lambda(t-s)}\|U(s,0)x\|$ • dissipation

$\lesssim \|C(t)U(t,0)x\|$ • uncertainty

Interp. $\lesssim \|U(s,0)x\|^{\theta}\,\|C(t)U(t,0)x\|^{1-\theta}$

$$\|U(\ell_m,0)x\| \lesssim \varepsilon\|U(\ell_{m+1},0)x\|^{\theta} + \varepsilon^{-\frac{\theta}{1-\theta}}\int_{\ell_m}^{\ell_{m+1}} \|C(t)U(t,0)x\|\,dt$$

$$\varepsilon^{\frac{\theta}{1-\theta}}\|U(\ell_m,0)x\| - \varepsilon^{\frac{1}{1-\theta}}\|U(\ell_{m+1},0)x\| \lesssim \int_{\ell_{m+1}}^{\ell_m} \|C(t)G(t)x\|\,dt$$

II Application to Diff. Ops. Ctd.

Op. $$\boxed{A(t)u := \mathcal{F}^{-1}(\mathfrak{a}(t,\cdot)\mathcal{F}u) = \sum_{|\alpha|_1\le m} a_\alpha(t)(-\mathrm{i})^\alpha \partial^\alpha u}$$

Evo. $$\boxed{U(t,s)u := \mathcal{F}^{-1}\left(\mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\cdot)\mathrm{d}\tau}\mathcal{F}u\right)}$$

Heat-Kernel Estimates $\Rightarrow$

Sufficient Cond. for
- Dissipation $\leftarrow$ Kernel. Est.
- Uncertainty $\leftarrow$ Logunov - Sereda

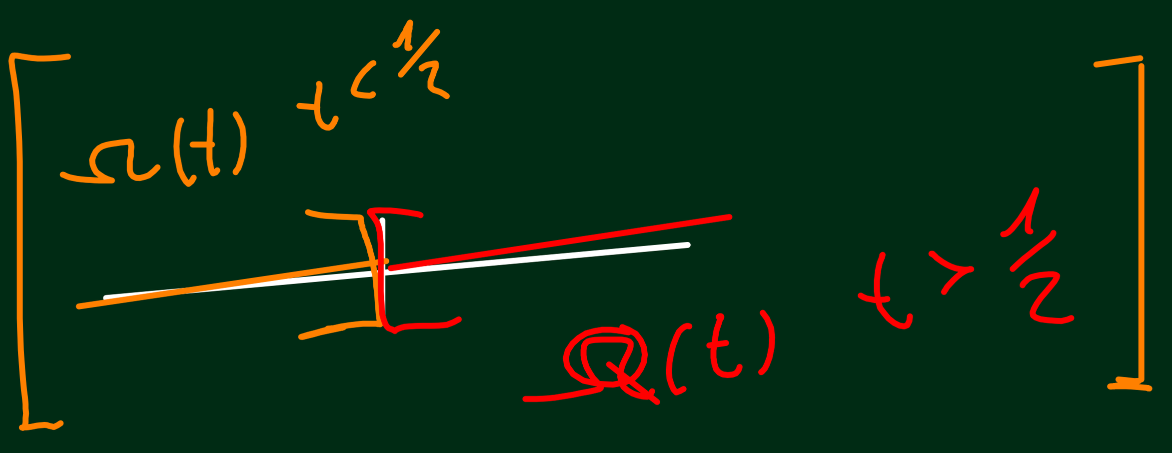
$(\Omega(t))$ $\qquad C(t) = 1_{\Omega(t)}$.

- measurable
- "thickness"

(1) $\Omega(t)$ unif. thick

$\forall t \quad |\Omega(t) \cap Q^d| \gtrsim |Q^d|$

(2) mean thick $\forall Q$ $\underbrace{\int_0^T |\Omega(t) \cap Q^d|}\gtrsim |Q^d|$



[BGST] Uniform thick $\to$ Observability

Observability $\leftrightarrow$ triv. $\leftrightarrow$ Uniform thick $\downarrow$ Mean Thick

Observability $\to$ Mean Thick (by contradict)

Mean Thick $\to$ Uniform thick: triv. $\Omega(t)$ [GST, BGST]

- Georg Nickel
- Pazy
- Lunardi

References (Links 2 go in zoom-chat)

[BGST] arXiv:2005.14503 [pdf, ps, other] math.FA math.AP math.OC
Observability and null-controllability for parabolic equations in L_p-spaces
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[GST] arXiv:1905.10285 [pdf, ps, other] math.FA math.AP math.OC doi 10.1137/19M1266769
Sufficient criteria and sharp geometric conditions for observability in Banach spaces
Authors: Dennis Gallaun, Christian Seifert, Martin Tautenhahn

[BEP-S] arXiv:1908.10603 [pdf, ps, other] math.AP
Geometric conditions for the null-controllability of hypoelliptic quadratic parabolic equations with moving control supports
Authors: Karine Beauchard, Michela Egidi, Karel Pravda-Starov

[BGabST] In preparation...