A Hermite-Hadamard type inequality with applications to the estimation of moments of convex functions of random variables

8ECM 2021- Approximation Theory and Applications

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- **Motivation**: applications of moment estimation in probability, statistics and statistical learning;
- A Hermite-Hadamard type inequality;
- Applications to the estimation of moments of convex functions of several random variables;
- We use some results previously obtained in [Gav2015];
- A particular case can be applied to estimate the sum of moments for identically distributed random variables.

The classical Hermite-Hadamard inequality

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

Generalized Theorem

Theorem ([Gav2015], Corollary 3.1)

Let $f \in C^1[0,1]$ and $w : [0,1] \to \mathbb{R}_+$ be a probability density function which is symmetric with respect to the midpoint $\frac{1}{2}$. If $|f'|$ is convex on $[0,1]$, then the following inequality holds:

$$\left| \int_0^1 f(x)w(x)dx - f\left(\frac{1}{2}\right) \right| \leq \left(|f'(0)| + |f'(1)|\right) \int_{\frac{1}{2}}^1 \left(x - \frac{1}{2}\right) w(x)dx. \qquad (2)$$

Main Result

Theorem

Let $D \subset \mathbb{R}^n$ be a convex and compact domain and $w : D \to \mathbb{R}_+$ be a probability density function with support D, i.e., $\int_D w(x)dx = 1$. Assume that $f : D \to \mathbb{R}$ is a twice continuously differentiable function ($f \in C^2(D)$), convex on D. Let $u \in D$ and $g_u : [0,1] \to \mathbb{R}$ be given by

$$g_u(t) = \int_D w(x) \frac{d}{dt}[f((1-t)u + tx)]dx. \tag{3}$$

If g_u' is convex on $[0,1]$, then the following inequality holds

$$M_{1/2}^u - \frac{1}{8}(M_0^u + M_1^u) \leq \int_D w(x)f(x)dx - f(u) \leq M_{1/2}^u + \frac{1}{8}(M_0^u + M_1^u), \tag{4}$$

$$\begin{aligned} M_0^u &= \int_D w(x)(x-u)^T \nabla^2 f(u)(x-u)dx, \\ M_1^u &= \int_D w(x)(x-u)^T \nabla^2 f(x)(x-u)dx, \\ M_{1/2}^u &= \int_D w(x)(x-u)^T \nabla f\left(\frac{x+u}{2}\right) dx. \end{aligned}$$

Main Theorem. Proof.

We define

$$I_u = \int_D w(x)f(x)dx - f(u).$$

It is easy to see that

$$I_u = \int_0^1 g_u(t)dt$$

We note that $g_u(t)$ can also be written in the form

$$g_u(t) = \int_D w(x)(x-u)^T \nabla f((1-t)u + tx)dx. \tag{5}$$

From (5), we also have

$$g_u'(t) = \int_D w(x)(x-u)^T \nabla^2 f((1-t)u + tx)(x-u)dx. \tag{6}$$

Since $f \in C^2(D)$ is a convex function, it follows from (6) that $g_u'(t) \geq 0$ and therefore

$$g_u'(0) = |g_u'(0)| = M_0^u \text{ and } g_u'(1) = |g_u'(1)| = M_1^u.$$

Since $g_u'(t) = |g_u'(t)|$ is convex on $[0, 1]$, we can use Theorem 1 together with the above identities as well as $g_u\left(\frac{1}{2}\right) = M_{1/2}^u$ and be led to the desired conclusion.

An application to the estimation of sum of moments of random variables

Let $p \in \mathbb{N}^*$, $v = (v_1, ..., v_n)^T \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$ convex and compact and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x) = \sum_{i=1}^{n} (x_i - v_i)^{2p}.$$

Clearly, f is convex and we have

$$\begin{aligned} \nabla f(x) &= 2p\left((x_1 - v_1)^{2p-1}, ..., (x_n - v_n)^{2p-1}\right)^T, \\ \nabla^2 f(x) &= 2p(2p-1)\mathrm{diag}\left((x_1 - v_1)^{2p-2}, ..., (x_n - v_n)^{2p-2}\right), \end{aligned}$$

A simple calculation gives

$$g_u'(t) = 2p(2p-1)\sum_{i=1}^{n}\int_D w(x)\left((1-t)u_i + tx_i - v_i\right)^{2p-2}(x_i-u_i)^2\,dx \tag{7}$$

From (7), it is straightforward to see that $g_u'(t)$ is convex for $t \in [0,1]$ and we have

$$M_0^u = 2p(2p-1)\sum_{i=1}^{n}(u_i-v_i)^{2p-2}\int_D w(x)(x_i-u_i)^2\,dx, \tag{8}$$

$$M_1^u = 2p(2p-1)\sum_{i=1}^{n}\int_D w(x)(x_i-v_i)^{2p-2}(x_i-u_i)^2\,dx, \tag{9}$$

$$M_{1/2}^u = \frac{p}{2^{2(p-1)}}\sum_{i=1}^{n}\int_D w(x)(x_i-u_i)(x_i+u_i-2v_i)^{2p-1}\,dx. \tag{10}$$

Consider now the following:

a) $X_1, ..., X_n$ with compact and convex support D,
b) n random variables identically distributed as the (generic) random variable X_0
c) $E[X_i]$ the expectation of the random variable X_i
d) $VAR[X_i]$ the variance of the random variable X_i
e) $\mu = E[X_i], i = \overline{0, n}$
f) $v = (v_1, ..., v_n)^T \in \mathbb{R}^n$
g) $u = \mu e$, with $e = (1, ..., 1)^T$.

Then, we obtain:

$$M_0^\mu = 2p(2p-1)VAR[X_0]\sum_{i=1}^{n}(v_i - \mu)^{2p-2}, \quad (11)$$

$$M_1^\mu = 2p(2p-1)\sum_{i=1}^{n} E\left[(X_0 - v_i)^{2p-2}(X_0 - \mu)^2\right], \quad (12)$$

$$M_{1/2}^\mu = \frac{p}{2^{2(p-1)}}\sum_{i=1}^{n} E\left[(X_0 - \mu)(X_0 + \mu - 2v_i)^{2p-1}\right]. \quad (13)$$

We can apply now the main result to obtain the following inequalities for the sum of the $2p$ moments of X_0 centered at v_1, ..., v_n

$$M_{1/2}^{\mu} - \frac{1}{8}(M_0^{\mu} + M_1^{\mu}) \leq \sum_{i=1}^{n} \left(E\left[(X_0 - v_i)^{2p} \right] - (\mu - v_i)^{2p} \right) \leq M_{1/2}^{\mu} + \frac{1}{8}(M_0^{\mu} + M_1^{\mu}). \quad (14)$$

In particular for $\gamma \in \mathbb{R}$ and $v = \gamma e$ we obtain that

$$E\left[(X_0-\gamma)^{2p}\right] - (\mu-\gamma)^{2p} \geq \frac{p}{2^{2(p-1)}} E\left[(X_0-\mu)(X_0+\mu-2\gamma)^{2p-1}\right] - \frac{p(2p-1)}{4}\left(E\left[(X_0-\gamma)^{2p-2}(X_0-\mu)^2\right] + (\gamma-\mu)^{2p-2} VAR[X_0]\right), \quad (15)$$

$$E\left[(X_0-\gamma)^{2p}\right] - (\mu-\gamma)^{2p} \leq \frac{p}{2^{2(p-1)}} E\left[(X_0-\mu)(X_0+\mu-2\gamma)^{2p-1}\right] + \frac{p(2p-1)}{4}\left(E\left[(X_0-\gamma)^{2p-2}(X_0-\mu)^2\right] + (\gamma-\mu)^{2p-2} VAR[X_0]\right) \quad (16)$$

These are lower and upper bounds for the difference between of the $2p$ moment centered at γ and the $2p$-power of the difference between the mean and γ.

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- Future work: relaxing the convexity assumption on $g_u'(t)$.

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