GCD Results on Semiabelian Varieties and a Conjecture of Silverman

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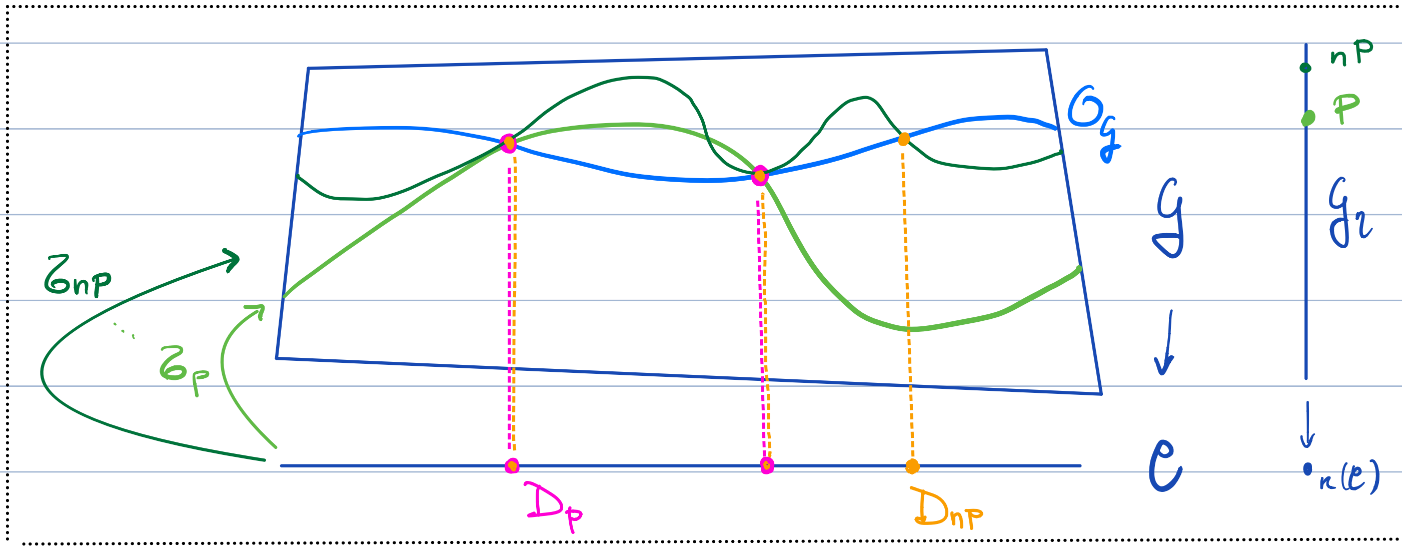

Geometric divisibility sequence [Silverman] $k = \overline{\mathbb{Q}}$

$\mathcal{G} \to C$ group scheme over non-sing. proj. curve $\mathcal{C}/k$

$G/k(C)$ generic fiber of $\mathcal{G}$

Assume G is IRREDUCIBLE, COMMUTATIVE, no UNIPOTENT PART

$\sigma_P : C \to \mathcal{G}$ section corresponding to $P \in G$

$O : C \to \mathcal{G}$ identity section

def: D_P divisor on $\mathcal{C}$ associated to $\sigma_P^* \mathcal{I}_O$

A geometric divisibility sequence is a seq. of divisor $(D_{nP})_{n \geq 1}$

↳ $n \mid m \Rightarrow D_{nP} \prec D_{nQ}$

EXAMPLES

1. $\mathcal{G} = \mathbb{G}_m \to \mathbb{P}^1$ $\qquad P = \frac{a(t)}{b(t)} \in G = \mathbb{G}_m(k(T)) = k(T)^\times$

$$\operatorname{supp} D_P = \{ t : a(t) = b(t) \}$$
$$\operatorname{supp} D_{nP} = \{ t : a^n(t) = b^n(t) \}$$

$m \mid n \Rightarrow D_{mP} \leq D_{nP}$ $\qquad$ in general $\operatorname{supp} D_{nP}$ NOT BOUNDED

2. $\mathcal{G} = \mathcal{E} \to \mathbb{P}^1$ elliptic scheme over $\mathbb{P}^1$

$P \in G = E_{/k(T)} \longleftrightarrow (x_P, y_P) \qquad x_P, y_P \in k(T)$

$$x_P = \frac{A_P}{\mathcal{D}_P^2} \qquad A_P, \mathcal{D}_P \in k[T]$$

$D_P = \frac{1}{2} \operatorname{div}(x_P)_\infty = \operatorname{div}(\mathcal{D}_P)_0$

$m \mid n \Rightarrow \mathcal{D}_{mP} \mid \mathcal{D}_{nP}$ as polinomials

OSS: P not torsion $\qquad \operatorname{supp} D_{nP}$ not bounded

SILVERMAN CONJECTURE (function field version)

Assume

ⓐ G has dimension $\dim_{k(C)} G \geq 2$

ⓑ The subgroup generated by P is $\mathbb{Z}$-dense in G

Then:

$$\boxed{D_{nP} = D_P \quad \text{for infinitely many } n}$$

MOTIVATION: result of Bugeaud - Corvaja - Zannier in $\mathbb{G}_m^2/\mathbb{Z}$

- $G = \mathbb{G}_m^2$ AILON - RUDNICK '04 $\rightsquigarrow$ $\gcd(a^n - 1, b^n - 1) \mid c$
  OSTAFE '16 general case

INDEP. OF n

- $G = E_1 \times E_2$ ell curves

  E_1, E_2 ISOTRIVIAL $\rightarrow$ Silverman '04

  general case $\rightarrow$ Ghioca - Xia - Tucker '18

THEOREM (Barroero - Capuano - T.)

Conjecture holds for $G = A \times \mathbb{G}_m^\ell$, A ab variety

i.e. for every $P \in G/_{k(C)}$ s.t. $\overline{\langle P \rangle}^{Zar} = G$

$D_{nP} = D_P$ for infinitely many $n \geq 1$

Moreover $\boxed{\exists \text{ effective divisor } D \text{ on } C \text{ s.t. } D_{nP} \leq D \text{ for every } n \geq 1}$

EXAMPLE $\mathcal{G} = \mathcal{E} \times \mathbb{G}_m \longrightarrow \mathcal{C} = \mathbb{P}^1$

$P = (Q, f) \in E \times \mathbb{G}_m /_{k(C)} = G$

$D_P = GCD(D_Q, D_f)$

$\exists\ D \in Div(C)$ s.t. $D_{nP} = GCD(D_{nQ}, D_{nf}) \leq D \quad \forall n \geq 1$
