Fractional dissipations in fluid dynamics: the surface quasigeostrophic equation

Maria Colombo

EPFL

EPFL SB, Institute of Mathematics

June 24th, 2021

ECM 2021, Nonlocal operators and related topics (MS - ID 55)

Table of contents

The SQG equation

The surface quasigeostrophic system (SQG) is

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = -(-\Delta)^{\frac{1}{2}} \theta \\ u = \mathcal{R}^{\perp} \theta := \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} \theta \,. \end{cases}$$

Here $\theta : \mathbb{R}^2 \times (0, \infty) \to \mathbb{R}$ represents the temperature and $u : \mathbb{R}^2 \times (0, \infty) \to \mathbb{R}^2$ the velocity.
We are interested in the Cauchy problem

$$\theta(\cdot, 0) = \theta_0 \,.$$

It is an advection-diffusion equation, with an incompressible vector field $\operatorname{div} u = 0$. More in general, we consider a fractional dissipation: $(-\Delta)^{\alpha} \theta = \mathcal{F}^{-1}(|\xi|^{2\alpha} \widehat{\theta})$ for $\alpha \in (0, 1)$.

The SQG equation

The surface quasigeostrophic system (SQG) is

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = -(-\Delta)^\alpha\theta \\ u = \mathcal{R}^\perp\theta := \nabla^\perp(-\Delta)^{\frac{1}{2}}\theta\,. \end{cases}$$

Here $\theta : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}$ represents the temperature and $u : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}^2$ the velocity.
We are interested in the Cauchy problem

$$\theta(\cdot,0) = \theta_0\,.$$

It is an advection-diffusion equation, with an incompressible vector field $\operatorname{div} u = 0$. More in general, we consider a fractional dissipation: $(-\Delta)^\alpha\theta = \mathcal{F}^{-1}(|\xi|^{2\alpha}\widehat{\theta})$ for $\alpha \in (0,1)$.

Conservation laws

1. The Hamiltonian of the system is conserved, i.e. for $t > 0$

$$H(t) := \|\theta(t)\|^2_{\dot{H}^{-1/2}} + 2\int_0^t \|\theta(s)\|^2_{\dot{H}^{\alpha-1/2}}\, ds = H(0)\,.$$

2. The total energy is conserved, i.e. for $t > 0$

$$\mathcal{E}(t) := \|\theta(t)\|^2_{L^2} + 2\int_0^t \|(-\Delta)^{\alpha/2}\theta(s)\|^2_{L^2}\, ds = \mathcal{E}(0)\,.$$

3. Maximum principle:

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty} \qquad \text{for } t > 0\,.$$

Scaling

SQG-α obeys a scaling symmetry: if (θ, u) solves it, then also

$$\theta_\lambda(x,t) := \lambda^{2\alpha-1}\theta(\lambda x, \lambda^{2\alpha} t) \quad u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha} t)\,.$$

We can compute the scaling of controlled quantities:

$$\|\theta_\lambda\|_{L^\infty} = \lambda^{2\alpha-1}\|\theta\|_{L^\infty}\,,$$

$$\mathcal{E}[\theta_\lambda](t) = \lambda^{2\alpha-2}\mathcal{E}[\theta](t).$$

$\alpha = 1/2$ is critical for the best controlled quantity. We are interested in the supercritical regime $\alpha < 1/2$.

Analogies with Euler / Navier-Stokes

The (inviscid) SQG system is

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0 \\ u = \mathcal{R}^\perp \theta := \nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta \,. \end{cases}$$

(where $\omega = \operatorname{curl} u$ represents the vorticity of u).
[Constantin-Majda-Tabak '94] proposed inviscid SQG as a simplified model for 3d Euler

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \,, \end{cases}$$

describing the potential to form finite-time singularities. The proposed blow-up scenario was ruled out by [Cordoba '98].
Even when dissipation is added SQG is a simplified model for Navier-Stokes, with conserved total energy, and a similar scaling analysis.

Analogies with Euler / Navier-Stokes

The (inviscid) 2d Euler system is

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \\ u = \mathcal{R}^\perp \theta := \nabla^\perp (-\Delta)^{-1} \omega \,. \end{cases}$$

(where $\omega = \operatorname{curl} u$ represents the vorticity of u).
[Constantin-Majda-Tabak '94] proposed inviscid SQG as a simplified model for 3d Euler

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \,, \end{cases}$$

describing the potential to form finite-time singularities. The proposed blow-up scenario was ruled out by [Cordoba '98].
Even when dissipation is added SQG is a simplified model for Navier-Stokes, with conserved total energy, and a similar scaling analysis.

Analogies with Euler / Navier-Stokes

The (inviscid) SQG system is

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0 \\ u = \mathcal{R}^\perp \theta := \nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta \, . \end{cases}$$

(where $\omega = \mathrm{curl}\, u$ represents the vorticity of u).
[Constantin-Majda-Tabak '94] proposed inviscid SQG as a simplified model for 3d Euler

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \mathrm{div}\, u = 0 \, , \end{cases}$$

describing the potential to form finite-time singularities. The proposed blow-up scenario was ruled out by [Cordoba '98].
Even when dissipation is added SQG is a simplified model for Navier-Stokes, with conserved total energy, and a similar scaling analysis.

Notions of solutions

(i) Distributional solutions: $\theta \in L^2(\mathbb{R}^2 \times [0, +\infty))$, $\forall \varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$

$$\int \theta(\partial_t \varphi - (-\Delta)^\alpha \varphi + u \cdot \nabla \varphi)\, dx\, dt = -\int \theta_0(x) \varphi(x, 0)\, dx\,.$$

We can rewrite $\int (u\theta) \cdot \nabla \varphi\, dx\, dt = \frac{1}{2} \int \theta [\mathcal{R}^\perp \cdot, \nabla \varphi] \theta\, dx\, dt$. Global existence from $\theta_0 \in \dot{H}^{-1/2}$ [Resnick '95, Marchand '08], and even in the inviscid case from $\theta_0 \in L^p$ with $p > \frac{4}{3}$!

(ii) Leray - Hopf solutions: distributional solutions with global energy inequality for a.e. $t \geq 0$

$$\frac{1}{2} \int |\theta(t)|^2 \,\mathrm{d}x + \int_0^t \int |(-\Delta)^{\alpha/2} \theta|^2 \,\mathrm{d}x \,\mathrm{d}\tau \leq \frac{1}{2} \int |\theta_0|^2 \,\mathrm{d}x\,.$$

Existence was proved by [Leray '34].

(iii) Classical solutions: local existence, blow-up problem.

Notions of solutions

(i) Distributional solutions: $\theta \in L^2(\mathbb{R}^2 \times [0,+\infty))$, $\forall \varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$

$$\int \theta(\partial_t \varphi - (-\Delta)^\alpha \varphi + u \cdot \nabla \varphi)\, dx\, dt = -\int \theta_0(x)\varphi(x,0)\, dx\,.$$

We can rewrite $\int (u\theta) \cdot \nabla \varphi \, dx\, dt = \frac{1}{2} \int \theta [\mathcal{R}^\perp \cdot, \nabla \varphi] \theta \, dx\, dt$.
Global existence from $\theta_0 \in \dot{H}^{-1/2}$ [Resnick '95, Marchand '08], and even in the inviscid case from $\theta_0 \in L^p$ with $p > \frac{4}{3}$!

(ii) Leray - Hopf solutions: distributional solutions with global energy inequality for a.e. $t \geq 0$

$$\frac{1}{2}\int |\theta(t)|^2 \, \mathrm{d}x + \int_0^t \int |(-\Delta)^{\alpha/2}\theta|^2 \, \mathrm{d}x \, \mathrm{d}\tau \leq \frac{1}{2}\int |\theta_0|^2 \, \mathrm{d}x\,.$$

Existence was proved by [Leray '34].

(iii) Classical solutions: local existence, blow-up problem.

Notions of solutions

1. Distributional solutions: $\theta \in L^2(\mathbb{R}^2 \times [0,+\infty))$, $\forall \varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$

$$\int \theta(\partial_t \varphi - (-\Delta)^\alpha \varphi + u \cdot \nabla \varphi)\, dx\, dt = - \int \theta_0(x)\varphi(x,0)\, dx\,.$$

We can rewrite $\int (u\theta)\cdot \nabla\varphi\, dx\, dt = \frac{1}{2}\int \theta[\mathcal{R}^\perp\cdot, \nabla\varphi]\theta\, dx\, dt$.
Global existence from $\theta_0 \in \dot{H}^{-1/2}$ [Resnick '95, Marchand '08], and even in the inviscid case from $\theta_0 \in L^p$ with $p > \frac{4}{3}$!

2. Leray - Hopf solutions: distributional solutions with global energy inequality for a.e. $t \geq 0$

$$\frac{1}{2}\int |\theta(t)|^2\, \mathrm{d}x + \int_0^t \int |(-\Delta)^{\alpha/2}\theta|^2\, \mathrm{d}x\, \mathrm{d}\tau \leq \frac{1}{2}\int |\theta_0|^2\, \mathrm{d}x\,.$$

Existence was proved by [Leray '34].

3. Classical solutions: local existence, blow-up problem.

Notions of solutions

1. Distributional solutions: $\theta \in L^2(\mathbb{R}^2 \times [0, +\infty))$, $\forall \varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$

$$\int \theta(\partial_t \varphi - (-\Delta)^\alpha \varphi + u \cdot \nabla \varphi)\, dx\, dt = -\int \theta_0(x)\varphi(x, 0)\, dx\, .$$

We can rewrite $\int (u\theta) \cdot \nabla \varphi \, dx\, dt = \frac{1}{2} \int \theta [\mathcal{R}^\perp \cdot, \nabla \varphi] \theta \, dx\, dt$.
Global existence from $\theta_0 \in \dot{H}^{-1/2}$ [Resnick '95, Marchand '08], and even in the inviscid case from $\theta_0 \in L^p$ with $p > \frac{4}{3}$!

2. Leray - Hopf solutions: distributional solutions with global energy inequality for a.e. $t \geq 0$

$$\frac{1}{2} \int |\theta(t)|^2 \, \mathrm{d}x + \int_0^t \int |(-\Delta)^{\alpha/2} \theta|^2 \, \mathrm{d}x \, \mathrm{d}\tau \leq \frac{1}{2} \int |\theta_0|^2 \, \mathrm{d}x\, .$$

Existence was proved by [Leray '34].

3. Classical solutions: local existence, blow-up problem.

Regular solutions in the (sub)critical case

Theorem ([Constantin-Wu '99] for $\alpha > 1/2$, [Kiselev-Nazarov-Volberg '07] and [Caffarelli-Vasseur '10] for $\alpha = 1/2$)

Let $\alpha \geq 1/2$ and let $\theta_0 \in L^2(\mathbb{R}^2)$. Then there exists a smooth solution (θ, u) of SQG starting from θ_0.

For $\alpha < 1/2$ this is a fascinating open problem. It is known:

- eventual regularization [Silvestre '10, Dabkowski '11, Kiselev '11]: Solutions are smooth for t sufficiently large.
- $L^2 \to L^\infty$ [Constantin-Wu '08] Leray-Hopf solutions are bounded for $t > 0$, via the De Giorgi method.
- conditional regularity [Constantin-Wu '09, ...] e.g. $u \in L^\infty_t C^\gamma_x$ with $\gamma > 1 - 2\alpha \Rightarrow \theta \in C^\infty$.

[Buckmaster-Vicol-Shkoller '16] proved "distributional non-uniqueness" for $\alpha < 3/4$.

The singular set

For $\alpha < 1/2$, can we still say something about the singular set

$$\mathrm{Sing}\,\theta := \big\{(x,t) : \theta \text{ is not locally smooth around } (x,t)\big\}\ ?$$

Is it compact, is it still a null set?

For Navier-Stokes, it holds $\dim_{\mathcal{H}}(\mathrm{Sing}_{\,T}\, u) \le \frac{1}{2}$ [Leray '34] and even $\mathcal{P}^1(\mathrm{Sing}\, u) = 0$ [Scheffer, Caffarelli-Kohn-Nirenberg '82].

Table of contents

Main result

Theorem (C.-Haffter '20)

Let $\alpha > \alpha_0 := \frac{1+\sqrt{33}}{16} \approx 0.42$. *For any* $\theta_0 \in L^2(\mathbb{R}^2)$ *there exists a Leray-Hopf weak solution* (θ, u) *of SQG-*α *and a relatively closed set* $\mathrm{Sing}\,\theta$ *such that*

- $\theta \in C^\infty\big([\mathbb{R}^2 \times (0,\infty)] \setminus \mathrm{Sing}\,\theta\big)$,
- *for every* $t > 0$ $\mathrm{Sing}\,\theta \cap [\mathbb{R}^2 \times [t,\infty)]$ *is compact,*
- $\dim_{\mathcal{H}} \mathrm{Sing}\,\theta \le \frac{1}{2\alpha}\big(\frac{1+\alpha}{\alpha}(1-2\alpha)+2\big)$.

- In particular, θ is smooth almost everywhere;
- $\mathrm{Sing}\,\theta$ is compact if θ_0 is sufficiently regular to ensure local smooth existence;
- the partial regularity holds for every "suitable weak solution".

Main result

Theorem (C.-Haffter '20)

Let $\alpha > \alpha_0 := \frac{1+\sqrt{33}}{16} \approx 0.42$. For any $\theta_0 \in L^2(\mathbb{R}^2)$ there exists a Leray-Hopf weak solution (θ, u) of SQG-α and a relatively closed set $\mathrm{Sing}\,\theta$ such that

- $\theta \in C^\infty\left([\mathbb{R}^2 \times (0,\infty)] \setminus \mathrm{Sing}\,\theta\right)$,
- *for every* $t > 0$ $\mathrm{Sing}\,\theta \cap [\mathbb{R}^2 \times [t,\infty)]$ *is compact,*
- $\dim_{\mathcal{H}} \mathrm{Sing}\,\theta \le \frac{1}{2\alpha}\left(\frac{1+\alpha}{\alpha}(1-2\alpha)+2\right)$.

- In particular, θ is smooth almost everywhere;
- $\mathrm{Sing}\,\theta$ is compact if θ_0 is sufficiently regular to ensure local smooth existence;
- the partial regularity holds for every "suitable weak solution".

ε-regularity Theorem

The result is a corollary of

Theorem (ε-regularity)

Let $\alpha > \alpha_0$,and $p := \frac{1+\alpha}{\alpha}$. Set $\beta := \frac{1}{2\alpha}(p(1-2\alpha)+2)$. Let (θ, u) be a suitable weak solution of SQG-α with

$$\frac{\|\theta\|_{L^\infty}^{p-2}}{r^\beta} \int_{t-r}^{t+r} \int_{B_{\|u\|_{L^\infty} r}(x)} |\nabla^\alpha \theta|^2 \, dz \, ds \leq \varepsilon(\alpha) \, .$$

Then θ is smooth on $B_{\frac{r^{1/(2\alpha)}}{8}}(x) \times [t - r/8, t + r/8]$.

- L^∞ norms are on $\mathbb{R}^2 \times [t-r, t+r]$.
- The choice of β is determined by scaling invariance.

ε-regularity Theorem

The result is a corollary of

Theorem (ε-regularity)

Let $\alpha > \alpha_0$,and $p := \frac{1+\alpha}{\alpha}$. Set $\beta := \frac{1}{2\alpha}(p(1-2\alpha)+2)$. Let (θ, u) be a suitable weak solution of SQG-α with

$$\frac{\|\theta\|_{L^\infty}^{p-2}}{r^\beta} \int_{t-r}^{t+r} \int_{B_{\|u\|_{L^\infty} r}(x)} |\nabla^\alpha \theta|^2 \, dz \, ds \le \varepsilon(\alpha) \, .$$

Then θ is smooth on $B_{\frac{r^{1/(2\alpha)}}{8}}(x) \times [t - r/8, t + r/8]$.

- L^∞ norms are on $\mathbb{R}^2 \times [t-r, t+r]$.
- The choice of β is determined by scaling invariance.

Comments on the statement

The statement is not precise for two reasons:

- $\|u\|_{L^\infty}$ is not under control since in the limiting case CZ reads as $\|u\|_{BMO} \leq \|\theta\|_{L^\infty}$. It can be solved by replacing $\|u\|_{L^\infty} r$ with $\|u\|_{L^q} r^{1-\frac{1}{\alpha q}}$ for arbitrarily large q.
- When writing
$$\int_{B_R} |\nabla^\alpha \theta|^2 \, dz$$
we really mean a localized quantity that involves the Caffarelli-Silvestre extension θ^* of θ
$$\int_{B_R \times [0,R]} y^b |\overline{\nabla} \theta^*|^2 \, dz \, dy.$$

Parabolic geometry

The parabolic cylinders are defined to respect the scaling of the equation

$$Q_r(x,t) = B_r(x) \times (t - r^{2\alpha}, t]$$

For $\alpha < \frac{1}{2}$

$$\operatorname{diam} Q_r(x,t) = \sqrt{r^{4\alpha} + (2r)^2} \lesssim r^{2\alpha}$$

and hence at scale r we work with $Q_{r^{1/(2\alpha)}}(x,t)$.

Is the dimension estimate optimal?

An estimate on the dimension of the singular set is based on

1. a globally bounded quantity in the form of a spacetime integral;
2. an ε-regularity criterion that involves in its smallness assumption a localized version of this integral quantity (on a spacetime set of diameter $\sim r$, such as $Q_{r^{1/(2\alpha)}}$)

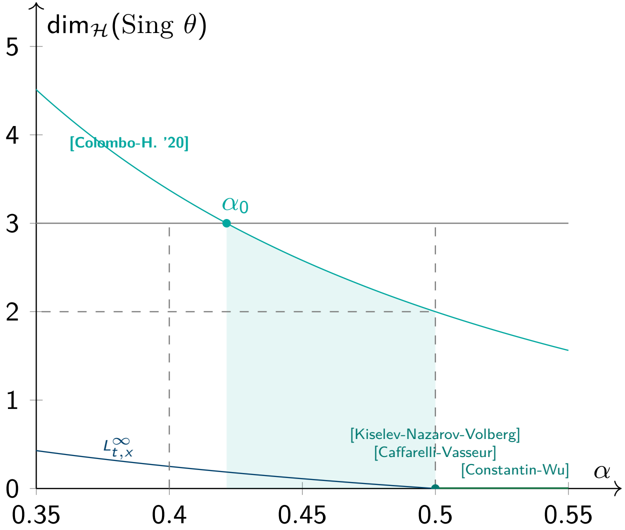
Then

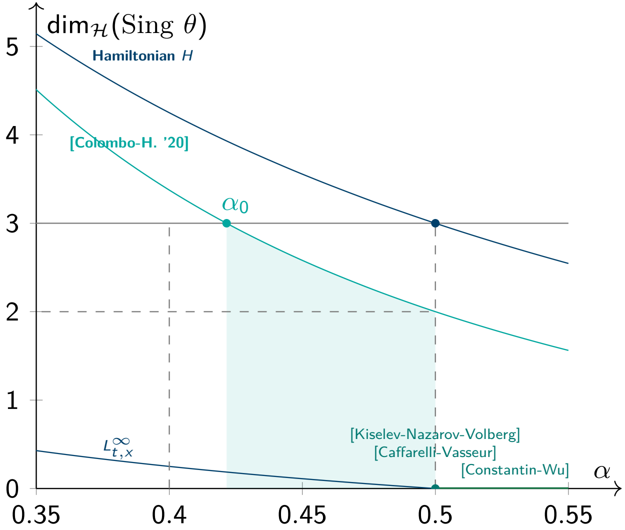
$$\dim_{\mathcal{H}} \mathrm{Sing}\ \theta = \text{scaling of this integral quantity on } Q_{r^{1/(2\alpha)}}\ .$$

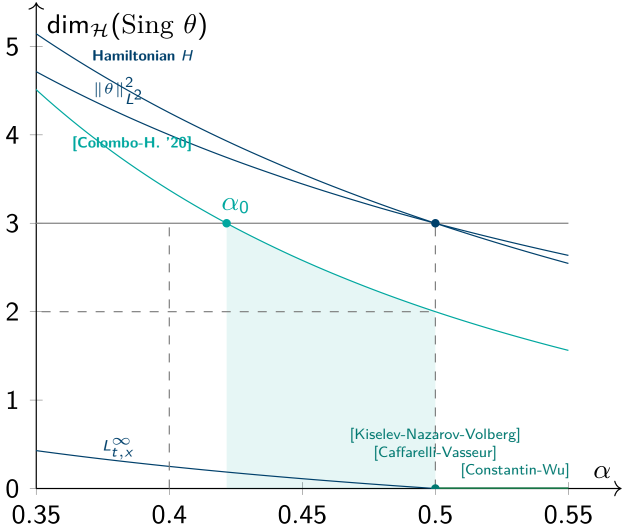
Conjecture (C.-Haffter '20)

Any suitable weak solution of SQG-α satisfies

$$\dim_{\mathcal{H}} \mathrm{Sing}\ \theta \leq \frac{2(1-\alpha)}{\alpha}\ .$$


$\dim_{\mathcal{H}}(\mathrm{Sing}\ \theta)$
5
4
3
2
1
0
[Colombo-H. '20]
α_0
$L^\infty_{t,x}$
[Kiselev-Nazarov-Volberg]
[Caffarelli-Vasseur]
[Constantin-Wu]
α
0.35
0.4
0.45
0.5
0.55


$\dim_{\mathcal{H}}(\mathrm{Sing}\ \theta)$
Hamiltonian H
[Colombo-H. '20]
α_0
$L^\infty_{t,x}$
[Kiselev-Nazarov-Volberg]
[Caffarelli-Vasseur]
[Constantin-Wu]
α
5
4
3
2
1
0
0.35
0.4
0.45
0.5
0.55


$\dim_{\mathcal{H}}(\mathrm{Sing}\ \theta)$
Hamiltonian H
$\|\theta\|^2_{L^2}$
[Colombo-H. '20]
α_0
$L^\infty_{t,x}$
[Kiselev-Nazarov-Volberg]
[Caffarelli-Vasseur]
[Constantin-Wu]
α
5
4
3
2
1
0
0.35
0.4
0.45
0.5
0.55

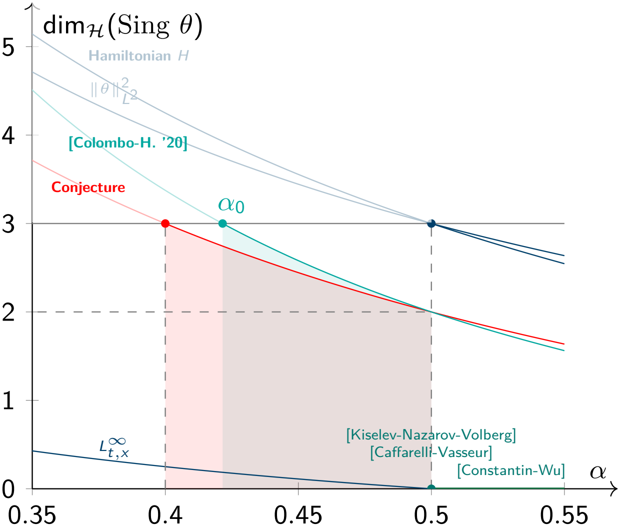

$\dim_{\mathcal{H}}(\text{Sing } \theta)$
5
4
3
2
1
0
Hamiltonian H
$\|\theta\|_{L^2}^2$
[Colombo-H. '20]
Conjecture
α_0
$L^\infty_{t,x}$
[Kiselev-Nazarov-Volberg]
[Caffarelli-Vasseur]
[Constantin-Wu]
α
0.35
0.4
0.45
0.5
0.55

Table of contents

Excess decay

We now define the "excess" of a suitable weak solution

$$E(\theta, u; x, t, r) := \left(\fint_{Q_r(x,t)} |\theta - (\theta)_{Q_r(x,t)}|^p \, dz\, ds \right)^{\frac{1}{p}} + \left(\fint_{Q_r(x,t)} |u - [u]_{B_r(x)}|^p \, dz\, ds \right)^{\frac{1}{p}} + tails\ldots$$

Theorem (Excess decay)

Let $\alpha \in (0,1)$ and $p > \frac{1+\alpha}{\alpha}$. For any $\gamma \in (0, 2\alpha(1-\frac{1}{p}))$ there exists $\varepsilon_0 \in (0,1)$ and $\mu_0 \in (0, \frac{1}{2})$ s.t. if (θ, u) is a suitable weak solution of SQG-α satisfying

- $[u(s)]_{B_r(x)} = 0$ *for all* $s \in [t - r^{2\alpha}, t]$,
- $E(\theta, u; x, t, r) \leq r^{1-2\alpha}\varepsilon_0$,

then the excess decays at scale μ_0 with rate γ, that is

$$E(\theta, u; x, t, \mu_0 r) \leq \mu_0^{\gamma} E(\theta, u; x, t, r) .$$

Main ideas of the proof

The proof has two parts:

1. Excess decay argument with 2 new ingredients.

   To prevent the lack of compactness of the local EI, we perform energy estimates of nonlinear type controlling $|\theta|^{p-1}$ $p > 3$. Hence we use L^∞ bound and a new notion of suitable weak solution.

   Change of variable along the flow to set certain averages of u to 0, since we lack other controls on the averages of the velocity.

2. reach initial smallness of the excess. Strategies:

   We pass from an L^p-based excess to a differential quantity via a nonlinear Poincaré inequality of parabolic type.

   We need again to control the effect of the "flow", which guarantees zero-average assumption by enlarging spacetime cylinders to contain the effect of the flow.

Main ideas of the proof

The proof has two parts:

1. Excess decay argument with 2 new ingredients.

   To prevent the lack of compactness of the local EI, we perform energy estimates of nonlinear type controlling $|\theta|^{p-1}$ $p > 3$. Hence we use L^∞ bound and a new notion of suitable weak solution.

   Change of variable along the flow to set certain averages of u to 0, since we lack other controls on the averages of the velocity.

2. reach initial smallness of the excess. Strategies:

   We pass from an L^p-based excess to a differential quantity via a nonlinear Poincaré inequality of parabolic type.

   We need again to control the effect of the "flow", which guarantees zero-average assumption by enlarging spacetime cylinders to contain the effect of the flow.

The end

Thank you for your attention!