The Bernstein technique for integro-differential equations

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8th European Congress of Mathematics
20-26 June 2021, Portorož (Slovenia)

Session Nonlocal Operators and Related Topics

Joint work with Serena Dipierro & Enrico Valdinoci (arXiv)

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- The <u>local</u> Bernstein technique:

$$\underline{\varphi := \eta^2 (\partial_e u)^2 + \sigma u^2}, \quad (\eta = \text{cut-off}, e \in \mathbb{R}^n, \sigma \in \mathbb{R}_+)$$

$$\Rightarrow \quad -\Delta \varphi \leq 2\eta^2 \partial_e u (-\Delta) \partial_e u + 2\sigma u (-\Delta) u$$

if $\sigma = \sigma(n)$ is large enough (this holds $\forall$ fc'n u)

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Now, if $\Delta u = 0$ in $B_1 \subset \mathbb{R}^n$ then $-\Delta \varphi \leq 0$ in B_1

$\Downarrow$ Maximum principle $(\eta_{|B_{1/2}} \equiv 1)$

$$\sup_{B_{1/2}} |\partial_e u|^2 \leq \sup_{\partial B_1} \varphi \leq \sigma \, \|u\|^2_{L^\infty(B_1)}.$$

INTERIOR GRADIENT ESTIMATE

- The auxiliary fc'n $\varphi = \eta^2 (\partial_e u)^2 + \sigma u^2$ also works for fully nonlinear elliptic equations $F(D^2 u) = 0$.
- More sophisticated auxiliary fc'ns work for the prescribed mean curvature eq'n.

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■ We extend these results on the quadratic auxiliary fc'n

for
- the fractional Laplacian
- integro-diff. op'rs with general kernels

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■ We extend these results on the <u>quadratic auxiliary fc'n</u>

for
- <u>the fractional Laplacian</u>
- <u>integro-diff. op'rs with general kernels</u>

$\rightsquigarrow$ results for fully nonlinear integro-diff. eq'ns.

■ RESULTS
- eq'ns with extension pb. $\longrightarrow$ 1st & one-sided 2nd derivative estimates
- eq'ns without extension $\longrightarrow$ 1st derivative estimates

(A) PUCCI-TYPE EQ'NS WITH EXTENSIONS

$$\mathcal{L}_A u(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|A(x-y)|^{n+2s}}\, dy \qquad (0<s<1,\ \lambda \cdot Id \leq A \leq \Lambda \cdot Id)$$

EQ'N: $\sup_A \left(\mathcal{L}_A u(x) - g_A(x) \right) = 0$ in $B_1 \subset \mathbb{R}^n$ (1)

given fc'ns

Ⓐ PUCCI-TYPE EQ'NS WITH EXTENSIONS

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Thm A $u \in (C^\infty \cap W^{2,\infty})(\mathbb{R}^n)$ solves (1) $\Longrightarrow$

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C_{n,\lambda,\Lambda} \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \sup_A \|g_A\|_{W^{1,\infty}(B_1)} \right)$$

&

$$\sup_{B_{1/2}} \partial_e^2 u \le C_{n,\lambda,\Lambda} \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \sup_A \|g_A\|_{W^{2,\infty}(B_1)} \right).$$

(B) PUCCI-TYPE EQNS FOR GENERAL INTEGRO-DIFF. OP'RS

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x) - u(y))\, K(x-y)\, dy,$$

$$\begin{cases} K(z) = K(-z), \quad \dfrac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \dfrac{\Lambda}{|z|^{n+2s}}, \\ |z|\,|\nabla K(z)| + |z|^2\,|D^2 K(z)| \leq \Lambda\, K(z). \end{cases}$$

EQ'N: $$\sup_B \left(\mathcal{L}_{K_B} u(x) - g_B(x) \right) = 0 \quad \text{in } B_1 \subset \mathbb{R}^n. \qquad (2)$$

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given fc'ns

Thm B $u \in (C^\infty \cap W^{1,\infty})(\mathbb{R}^n)$ solves (2) $\Rightarrow$

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C_{n,s,\lambda,\Lambda} \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \sup_B \|g_B\|_{W^{1,\infty}(B_1)} \right).$$

(c) OTHER FULLY NONLINEAR EQ'NS (+ INDEFINITE ORDER)

$$F(\mathcal{L}_1 u - g_1, \dots, \mathcal{L}_J u - g_J) = 0 \quad \text{in } B_1 \subset \mathbb{R}^n$$

with F convex & $\lambda \leq \sum_{j=1}^{J} \partial_{p_j} F(p) \leq \Lambda \quad \forall p \in \mathbb{R}^J$.

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(c1) $\mathcal{L}_j = \mathcal{L}_{K_j}$ $\rightsquigarrow$ 1^{st} derivative estimates.

(c2) $\mathcal{L}_j = \mathcal{L}_{\mu_j}$, $\mathcal{L}_{\mu_j} u(x) := \int_0^1 (-\Delta)^s u(x)\, d\mu_j(s)$ $\rightsquigarrow$

$\rightsquigarrow$ 1^{st} & one-sided 2^{nd} derivative estimates.

$\rightarrow$ **COROLLARY** (Obstacle problem) $\max\{(-\Delta)^s u, u - \phi\} = f$

- **PREVIOUS WORKS** : derivative estimates for fully nonlinear integro-diff. eqns in :
  - [Jakobsen – Karlsen '05] : Ishii-Lions doubling variables method
  - [Caffarelli – Silvestre '09] : Krylov-Safonov method ($C^{1,\alpha}$ est.)
  - [Barles-Chasseigne-Ciomaga-Imbert '12] : Ishii-Lions method

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→ Instead we use the quadratic auxiliary fcn.

↓

our method has been used by [Fernandez-Real & Jhaveri, Anal PDE] to extend derivative bounds for fractional obstacle pbs (1st result of [Athanasopoulos-Caffarelli '04])

• The Key inequality for the fractional Laplacian

PROP'N $\forall u \in (C^\infty \cap W^{1,\infty})(\mathbb{R}^n)$, $s \in (0,1)$,

$$(-\Delta)^s \left(\eta^2 (\partial_e u)^2 + \sigma u^2 \right) \leq 2\eta^2 \partial_e u \, (-\Delta)^s \partial_e u + 2\sigma \, u \, (-\Delta)^s u$$

if $\sigma \geq \sigma(n,\eta)$ is large enough.

• Easy PROOF using the extension problem.

→ Similar inequality replacing $\partial_e u \rightsquigarrow (\partial_e v)_+$

Take then $v = \partial_e u$

→ One-sided 2nd deriv. estimates

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Take then $v = \partial_e u$ $\Big\}$ $\rightarrow$ One-sided 2^{nd} deriv. estimates

OPEN PB 1 : Prove the Prop'n downstairs, in $\mathbb{R}^n$, without using the extension.

- We only know how to do it when $(-\Delta)^s u = 0$ (ideas later)

• The key inequality for integro-diff. op'rs.

LEMMA For every smooth $u:\mathbb{R}^n \to \mathbb{R}$, $K \in \mathcal{K}_2$, $E \in \mathbb{R}$ (an "error"):

$$\mathcal{L}_K\left(\eta^2(\partial_e u)^2 + \sigma u^2\right) \leq 2\eta^2 \partial_e u\, \mathcal{L}_K \partial_e u + 2\sigma\, u\, \mathcal{L}_K u + E \quad \text{at } x \in \mathbb{R}^n$$

$$\Updownarrow$$

$$2\int_{\mathbb{R}^n} \eta(x)(\eta(x)-\eta(y))\,\partial_e u(x)\,\partial_e u(y)\,K(x-y)\,dy$$

$$\leq \int_{\mathbb{R}^n} |\eta(x)\partial_e u(x) - \eta(y)\partial_e u(y)|^2 K(x-y)\,dy$$

$$+ \sigma \int_{\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dy + E.$$

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$$2\int_{\mathbb{R}^n}\eta(x)(\eta(x)-\eta(y))\,\partial_e u(x)\,\partial_e u(y)\,K(x-y)\,dy$$
$$\leq\int_{\mathbb{R}^n}|\eta(x)\partial_e u(x)-\eta(y)\partial_e u(y)|^2K(x-y)\,dy$$
$$+\sigma\int_{\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dy\;+E.$$

THM C $\forall\varepsilon>0\ \exists\,\sigma_\varepsilon=\sigma_\varepsilon(\varepsilon,\eta,n,s,\lambda,\Lambda)$ such that $\forall u:\mathbb{R}^n\to\mathbb{R}$

$$\mathcal{L}_K\left(\eta^2(\partial_e u)^2+\sigma_\varepsilon u^2\right)\leq 2\eta^2\partial_e u\,\mathcal{L}_K\partial_e u+2\sigma_\varepsilon\, u\,\mathcal{L}_K u+\varepsilon^2\|\partial_e u\|^2_{L^\infty(B_3)}$$

everywhere in B_2.

Since the same inequality will hold in any ball $B_{2\rho}(x)$ with an error computed in $B_{3\rho}(x)$, by scaling,
we are able to reabsorb the error $\|\partial_e u\|_{L^\infty(B_3)}$ and get an estimate for $\|\partial_e u\|_{L^\infty(B_1)}$. $\square$

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<u>OPEN PB 2</u> Does <u>Thm C</u> hold without the error term $+\,\varepsilon^2 \|\partial_e u\|^2_{L^\infty(B_3)}$?

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OPEN PB 2 Does Thm C hold without the error term $+\varepsilon^2\|\partial_e u\|^2_{L^\infty(B_3)}$?

OPEN PB 3 • In Thm B (1^{st} deriv. estimate for Bellman eq'n with general kernels), do one-sided second derivative estimates hold ?

• Does Thm C hold with $\partial_e u$ replaced by $(\partial_e v)_+$?

Ideas of PROOF of Thm C :

Need to prove

$$2\int_{\mathbb{R}^n} \eta(0)\big(\eta(0)-\eta(y)\big)\partial_e u(0)\,\partial_e u(y)\,K(y)\,dy$$

$$\le \int_{\mathbb{R}^n} \big|\eta(0)\partial_e u(0)-\eta(y)\partial_e u(y)\big|^2 K(y)\,dy + \delta\int_{\mathbb{R}^n} |u(0)-u(y)|^2 K(y)\,dy + E.$$

$$\text{LHS} = 2\int_{\mathbb{R}^n} \big(\eta(0)-\eta(y)\big)\,\eta(0)\,u_e(0)\,\partial_e\big(u(y)-u(0)\big)\,K(y)\,dy$$

$$= -2\int_{\mathbb{R}^n} \partial_e\Big(\big(\eta(0)-\eta(y)\big)K(y)\Big)\,\eta(0)\,u_e(0)\,\big(u(y)-u(0)\big)\,dy$$

$$= -2\int_{\mathbb{R}^n} \partial_e\Big(\big(\eta(0)-\eta(y)\big)K(y)\Big)\big(\eta(0)u_e(0)-\eta(y)u_e(y)\big)\big(u(y)-u(0)\big)\,dy$$

$$-2\int_{\mathbb{R}^n} \partial_e\Big(\big(\eta(0)-\eta(y)\big)K(y)\Big)\,\eta(y)\,\underbrace{u_e(y)\big(u(y)-u(0)\big)}_{=\,\partial_e\frac{(u(y)-u(0))^2}{2}}\,dy$$

$$= -2\int_{\mathbb{R}^n} \frac{\partial_e\big((\eta(0)-\eta(y))K(y)\big)}{K(y)} \left\{\big(\eta(0)u_e(0)-\eta(y)u_e(y)\big)\sqrt{K(y)}\right\}\left\{\big(u(y)-u(0)\big)\sqrt{K(y)}\right\}dy$$

$$+\int_{\mathbb{R}^n} \frac{\partial_e\left\{\eta(y)\,\partial_e\big((\eta(0)-\eta(y))K(y)\big)\right\}}{K(y)}\cdot|u(y)-u(0)|^2K(y)\,dy$$

all equalities up to this point ☺

$$= -2\int_{\mathbb{R}^n} \frac{\partial_e\big((\eta(0)-\eta(y))K(y)\big)}{K(y)} \Big\{ \big(\eta(0)u_e(0)-\eta(y)u_e(y)\big)\sqrt{K(y)} \Big\} \Big\{ (u(y)-u(0))\sqrt{K(y)} \Big\} dy$$

$$+ \int_{\mathbb{R}^n} \frac{\partial_e\Big\{\eta(y)\,\partial_e\big((\eta(0)-\eta(y))K(y)\big)\Big\}}{K(y)} \cdot |u(y)-u(0)|^2 K(y)\, dy$$

all equalities up to this point ☺

- This integral is fine, by C-S, since $\frac{1}{K(y)}\partial_e\big((\eta(0)-\eta(y))K(y)\big)$ is bounded.

$\eta(0)-\eta(y) = O(|y|)$

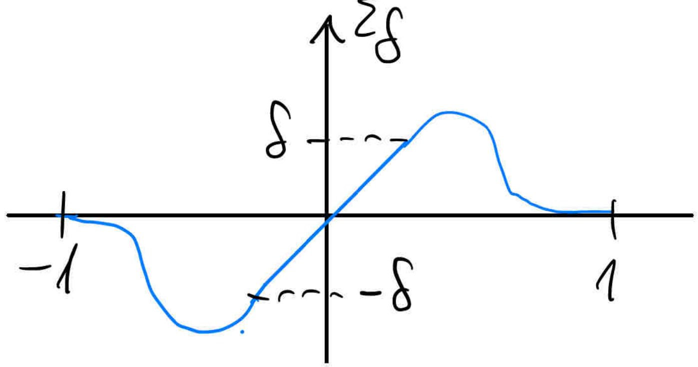
- this integral is not fine: the 1st factor is not bounded.

To prove Thm C (with the error term), we correct the initial LHS as follows:

$$\int_{\mathbb{R}^n} 2\eta(0)\big(\eta(0)-\eta(y)\big)\, u_e(0)\, u_e(y)\, K(y)\, dy$$

$$= \int_{\mathbb{R}^n} \Big\{ 2\eta(0)\big(\eta(0)-\eta(y)\big)\, u_e(0)\, u_e(y) + 2\eta(0)\, |u_e(0)|^2\, \nabla\eta(0)\cdot \vec{\xi}_\delta(y) \Big\} K(y)\, dy$$

where



$$\vec{\xi}_\delta(y) = \big(\xi_\delta(y_1), \ldots, \xi_\delta(y_n)\big)$$

This corrects the previous bad term, BVT gives two other terms:

$$\int_{\mathbb{R}^n} 2\eta(0)\big(\eta(0)-\eta(y)\big)\, u_e(0)\, u_e(y)\, K(y)\; dy$$

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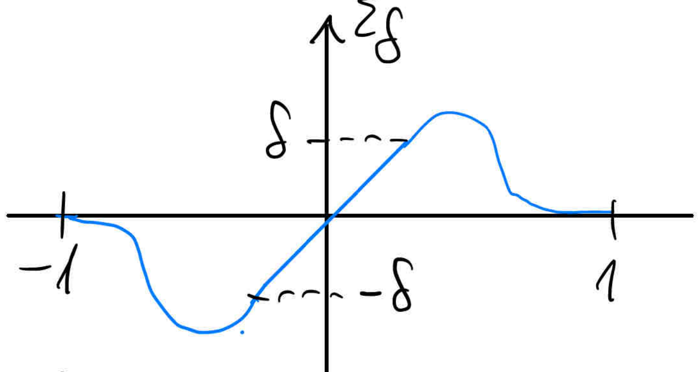

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$x=0$ will be later any point in B_1 !!

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$$+\, 2\big(\eta(y)-\eta(0)\big)\, u_e(0)\, u_e(y)\, \nabla\eta(0)\cdot\vec{\xi}_\delta(y)\, K(y)$$

& use

$$\int_{B_1} |y|\, |\vec{\xi}_\delta(y)|\, K(y)\, dy \le \varepsilon$$

if δ small $\blacksquare$

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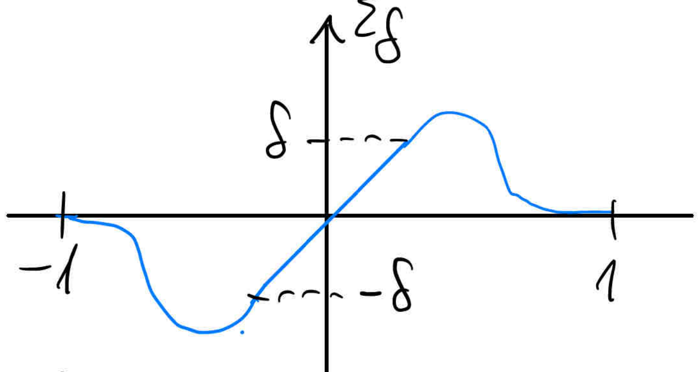

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THANKS FOR YOUR ATTENTION