Some Problems for a Mixed Type Equation Fractional Order with non-linear Loaded Term

Obidjon Abdullaev

obidjon.mth@gmail.com

V.I.Romanovskiy Institute of Mathematics , Tashkent, Uzbekistan

June 24, 2021

Table of Contents

1 Introduction
2 Problem formulation
- Non-local case
- Other problems
- Special cases

3 Conclusion

- Note, that with intensive research on problems of optimal control of the agro-economical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate BVPs for a loaded partial differential equations.
  Integral boundary conditions have various applications in thermo-elasticity, chemical engineering, population dynamics, etc. In this work we consider parabolic-hyperbolic type equation fractional order involving non-linear loaded term:

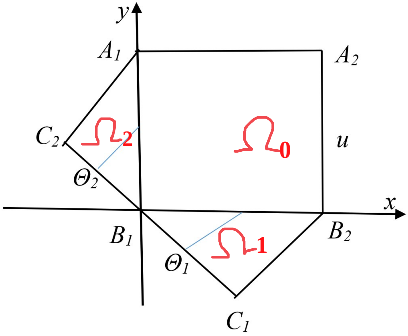
$$0 = \begin{cases} u_{xx} - {}_C D_{oy}^{\alpha} u + f_1(x, y; u(x, 0)), & \text{at } x > 0,\ y > 0 \\ u_{xx} - u_{yy} + f_2(x, y; u(x + y, 0)), & \text{at } x > 0,\ y < 0 \\ u_{xx} - u_{yy} + f_3(x, y; u_y(0, x + y)), & \text{at } x < 0,\ y > 0 \end{cases} \qquad (1)$$

where $f_i(\cdot)$ are given functions, and ${}_C D_{0y}^{\alpha}$ is Caputo operator:

$${}_C D_{0y}^{\alpha} f(y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-z)^{-\alpha} f'(z) dz, \quad 0 < \alpha < 1, \qquad (2)$$

Considered domain

- Let $\Omega \subset R^2$, be domain bounded with segments B_2A_2, A_2A_1 on the lines $x = l$, $y = h$ at $x > 0$, $y > 0$; and A_1C_2, C_2B_1 on the characteristics $x - y = l$, $x + y = 0$ of the Eq. (1) at $x > 0$, $y < 0$, also with the segments B_1C_1, C_1B_2 on the characteristics $y - x = h$, $x + y = 0$ of the Eq. (1) at $x < 0$, $y > 0$.
  We denote as Ω_0 parabolic part of the mixed domain Ω, and hyperbolic parts through Ω_1 at $x > 0$ and Ω_2 at $x < 0$.



Non-local problem

- In the domain Ω ($\Omega=\Omega_1\cup(B_1B_2)\cup\Omega_0\cup(B_1A_1)\cup\Omega_2,$) we will investigate following
  NL problem. To find a solution $u(x,y)$ of Eq.(1) from the class:
  W=$\left\{u\in C(\bar{\Omega})\cap C^1\left(\{\overline{\Omega}_2\setminus\overline{A_1C_2}\}\cup\{\overline{\Omega}_1\setminus\overline{C_1B_2}\}\right)\cap C^2(\Omega_2\cup\Omega_1);\right.$

$\left.u_{xx},\, {}_CD^{\alpha}_{oy}u\in C\left(\Omega_0\right)\right\}$;

satisfies boundary conditions:

$$u(l,t)=\varphi_1(y),\ \ 0\le y\le h; \tag{3}$$

$$\frac{d}{dx}u\left(\frac{x}{2};-\frac{x}{2}\right)=a_1(x)u_y(x,0)+a_2(x)u_x(x,0)+a_3(x)u(x,0)+$$
$$+a_4(x),\ \ 0\le x<l; \tag{4}$$

$$\frac{d}{dy}u\left(-\frac{y}{2};\frac{y}{2}\right)=b_1(y)u_x(0,y)+b_2(y)u_y(0,y)+b_3(y)u(0,y)+$$
$$+b_4(y),\ 0\le y<h; \tag{5}$$

and integral gluing condition:

$$\lim_{y\to+0} y^{1-\alpha}u_y(x,y)=\lambda_1(x)u_y(x,-0)+\lambda_2(x)u_x(x,-0)+$$

$$+\lambda_3(x)\int\limits_0^x r_1(t)u(t,0)dt+\lambda_4(x)u(x,0)+\lambda_5(x),\ 0<x<l \quad (6)$$

$$u_x(-0,y)=\mu_1(y)u_x(+0,y)+\mu_2(y)u_y(0,y)+\mu_3(y)\int\limits_0^y r_2(t)u(0,t)dt+$$

$$+\mu_4(y)u(0,y)+\mu_5(y),\ 0<y<h \quad (7)$$

where $a_i(x),\ b_i(y),\ \ (i=1,2,3,4),\ \lambda_j(x),\ \ \mu_j(y),\ (j=1,2,3,4,5)$, $\varphi_1(y)$ are given functions, such that $\sum\limits_{k=1}^{4}\lambda_k^2(x)\neq 0$, $\sum\limits_{k=1}^{3}a_k^2(x)\neq 0$, $\sum\limits_{k=1}^{3}b_k^2(x)\neq 0$ and $\sum\limits_{k=1}^{4}\mu_k^2(x)\neq 0$.

Remark.

The same problem with the local boundary and discontinuous gluing conditions for the Eq.(1) but without loaded term (i.e. for $f_i(x,y) \equiv 0 \, (i = 1,2,3)$) was investigated by B.Kadirkulov [1]. Formulated Problem in some special cases we will investigate with the same methods as in work [1], and a class of solution be expanded.

For instance

If one of the non-local conditions (4) and (5) changes to local condition (see fig.2), then the solution of the new problem just look in the class of functions:
$W_1 = \{u : u \in C(\bar{\Omega}) \cap C^1\left(\Omega_2 \cup B_1C_2\right) \cap C^2(\Omega_2 \cup \Omega_1);\ u_{xx}, {}_CD^{\alpha}_{oy}u \in C\left(\Omega_0\right)\}$ in case *a*),
and
$W_2 = \{u : u \in C(\bar{\Omega}) \cap C^1\left(\Omega_1 \cup B_1C_1\right) \cap C^2(\Omega_2 \cup \Omega_1);\ u_{xx}, {}_CD^{\alpha}_{oy}u \in C\left(\Omega_0\right)\}$ in case *b*).

Other problems

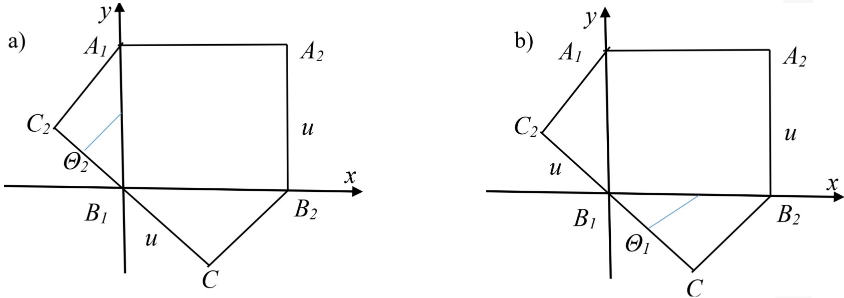


Figure: 2

The Method of investigation

On the certain conditions for the given functions, we can prove uniqueness of solution of the problem in cases *a*) and *b*), using by the method of integral energy. The existence of solution reduced to the Volterra and Fregholm types non-linear integral equations of the second kind respected to $u_y(0, y) = \tau_2'(y)$ and $u(x, 0) = \tau_1(x)$ accordingly.

Volterra type integral equations

We would like to note, that there are some problems for the Eq.(1), whose investigations will be reduced to the Volterra type non-linear integral equations (only), for instance:

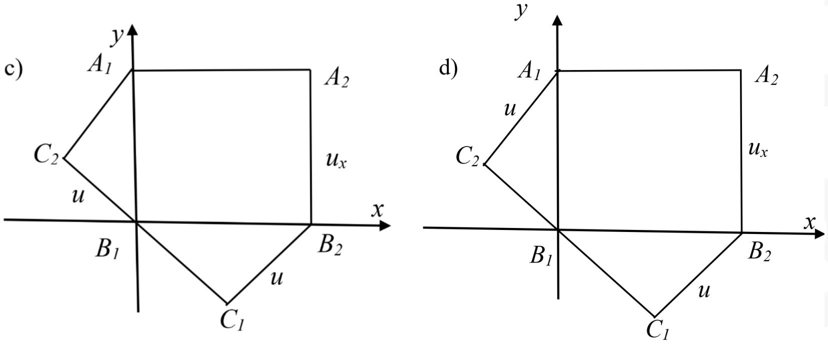


Figure: 3

In cases *c*). and *d*.) boundary condition (3) changed to the second boundary condition and the conditions (4) and (5) was replaced to the local conditions. In these cases, solution of the formulated problems, we are looking for in the class of functions:
$W_3 = \{u : u \in C(\bar{\Omega}) \cap C^2(\Omega_2 \cup \Omega_1);\ u_x \in C\left(\Omega_0 \cup \overline{A_2B_2}\right); u_{xx}, {}_C D_{oy}^{\alpha} u \in C\left(\Omega_0\right)\}$.
Considering solution of the Cauchy problem (in hyperbolic domains) and a solution of the first BVP for the parabolic equation (see Eq.(1)), applying boundary conditions (4) and (5), taking into

$$\lim_{y \to +0} y^{1-\alpha} u_y(x, y) = \nu_1^+(x),\ \ u_x(+0, y) = \nu_2^+(y),$$

$$u_x(-0, y) = \nu_2^-(y),\ \ u_y(x, -0) = \nu_1^-(x)$$

we will find main functional relation:

Main functional relations

$$(2a_1(x)+1)\nu_1^-(x) = (1-2a_2(x))\tau_1'(x) - 2a_3(x)\tau_1(x) -$$

$$-\frac{1}{2}\int_0^x f_2(\xi, x; \tau(\xi))d\xi - 2a_4(x), \qquad (8)$$

$$(2b_1(y)+1)\nu_2(y) = (1-2b_2(y))\tau_2'(y) - 2b_3(y)\tau_2(y) -$$

$$-\frac{1}{2}\int_0^y f_3(\xi, y; \tau_y(\xi))d\xi - 2b_4(y), \qquad (9)$$

$$\tau_1''(x) - \Gamma(\alpha)\nu_1^+(x) + f_1(x, 0; \tau_1(x)) = 0, \qquad (10)$$

$$\nu_2^+(y) = -\int_0^y K(y-t)\tau_2'(t)dt + \Phi(y), \qquad (11)$$

where $K(y-t)$ and $\Phi(y)$ known functions.

Based on the gluing conditions (see (6) and (7))

$$\nu_1^+(x) = \lambda_1(x)\nu_1^-(x) + \lambda_2(x)\tau_1'(x) + \lambda_3(x)\tau_1(x)dt + \lambda_4(x), \qquad (12)$$

$$\nu_2^-(y) = \mu_1(y)\nu_2^+(y) + \mu_2(y)\tau_2(y) + \mu_3(y),\ 0 < y < h, \qquad (13)$$

and considering (8)-(11), we receive an integral equation regarding $u(x,0) = \tau_1(x)$ and $u_y(0,y) = \tau_2'(y)$.
Assume that the given functions are sufficiently smooth functions, and $f_1(x, y, u(x,0)) \in C(\overline{\Omega}_0)$; $f_2(x, y, u(x+y, 0)) \in C(\overline{\Omega}_1)$; $f_3(x, y, u_y(0, x+y)) \in C(\overline{\Omega}_2)$, besides $f_i(x, y, u)\,(i = 1, 2, 3)$ satisfies Lipshits condition respected to the third argument, then we can prove unique solvability of the given integral equations

Reference

Bakhtiyor J. Kadirkulov *Boundary problems for mixed parabolic-hyperbolic equations with two lines of changing type and fractional derivative.*, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 57, pp. 17.

Thank You !