Stability and asymptotic properties of dissipative equations coupled with ordinary differential equations

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Outline

Motivation

Many problems from physics correspond to the coupling between a (dissipative) evolution equation and an ODE:

$$\begin{cases} U_t = \mathcal{A}U + MP, \text{ in } H, \\ P_t = BP + NU, \text{ in } X, \\ U(0) = U_0, P(0) = P_0, \end{cases} \tag{1}$$

where

- $\mathcal{A}$ is the generator of a C_0 semigroup in a Hilbert space H,
- B is a bounded operator from another Hilbert space X,
- $M : X \to H$, $N : H \to X$ bd operators.

Examples:

- dispersive medium models,
- generalized telegraph equations,
- Volterra integro-differential equations,
- cascades of ODE-hyperbolic systems.

Main questions

- Strong stability of the solution.
- Uniform Stability.
- Polynomial Stability.

The energy space

Introduce the (unbounded) operator $\mathbb{A}$ from $H \times X$ into itself as follows:

$$\mathbb{A}(U, P)^\top = \begin{pmatrix} \mathcal{A}U + MP \\ BP + NU \end{pmatrix}, \forall (U, P)^\top \in D(\mathbb{A}) = D(\mathcal{A}) \times X.$$

This allows to recast (1) as the Cauchy problem: Find $\mathrm{U} = (U, P)^\top$ s. t.

$$\begin{cases} \mathrm{U}_t = \mathbb{A}\mathrm{U} \text{ in } H \times X, \\ \mathrm{U}(0) = (U_0, P_0)^\top. \end{cases} \tag{2}$$

As

$$\mathbb{A}_0(U, P)^\top = \begin{pmatrix} \mathcal{A}U \\ 0 \end{pmatrix}, \forall (U, P)^\top \in D(\mathbb{A}).$$

generates a C_0-semigroup on $H \times X$ and $\mathbb{A} - \mathbb{A}_0$ is a bounded operator, a standard perturbation argument allows to conclude that $\mathbb{A}$ also generates a C_0-semigroup on $H \times X$.

Arendt-Batty/Lyubich-Vũ's thm

One simple way to prove the strong stability is to use the following

Theorem (Arendt-Batty/Lyubich-Vũ: Thm 1)

Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a bounded semigroup generated by A on X. Assume that no eigenvalues of A lies on the imaginary axis. If $\sigma(A)\cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is stable.

Since the resolvent of our operator is not compact (if $\dim X = +\infty$), we need to analyze the full spectrum on the imaginary axis.

W. Arendt and C. J. K. Batty.
Tauberian theorems and stability of one-parameter semigroups.
Trans. Amer. Math. Soc., 305(2):837–852, 1988.

Y. I. Lyubich and Q. P. Vũ.
Asymptotic stability of linear differential equations in Banach spaces. *Studia Math.*, 88(1):37–42, 1988.

Dissipativeness

To prove the boundedness property of the semigroup, we can use a criterion on the resolvent of $\mathbb{A}$, which may be a difficult task. A more restrictive condition, but satisfied in many applications, is to assume that $\mathbb{A}$ is dissipative, namely that

$$\Re(\mathbb{A}(U,P)^\top,(U,P)^\top)_{H\times X} \leq 0, \forall (U,P)^\top \in D(\mathcal{A}) \times X. \tag{3}$$

Indeed in such a case, by Lumer-Phillips' theorem it generates a C_0-semigroup of contractions on $H \times X$.
Therefore the use of Theorem 1 is reduced to the analysis of $\rho(A) \cap \imath\mathbb{R}$.

The point spectrum: one criterion

Lemma (Le 2)

If

$$\Re(\mathbb{A}(U,P)^\top,(U,P)^\top)_{H\times X} \lesssim -\|P\|_X^2, \forall (U,P)^\top \in D(\mathcal{A})\times X \quad (4)$$

holds, then for all $\xi \in \mathbb{R}$, *one has*

$$\ker(\imath\xi\mathbb{I}-\mathbb{A}) = \{(U,0)^\top \mid U \in \ker N \cap \ker(\imath\xi\mathbb{I}-\mathcal{A})\}. \quad (5)$$

In particular $\sigma_p(\mathbb{A})\cap \imath\mathbb{R}=\emptyset$ *iff*

$$\ker N \cap \ker(\imath\xi\mathbb{I}-\mathcal{A}) = \{0\}, \forall \xi \in \mathbb{R}. \quad (6)$$

Pf. $(U,P)^\top \in \ker(\imath\xi\mathbb{I}-\mathbb{A})$ iff

$$\begin{cases} \imath\xi U - \mathcal{A}U - MP = 0, \\ \imath\xi P - BP - NU = 0. \end{cases}$$

By (4), $P=0$, hence $(\imath\xi-\mathcal{A})U=0$ and $NU=0$.

Closedness of the range

Corollary

Let (4) *be satisfied and suppose given a bd operator* $C : X \to H$. *If* $\xi \in \mathbb{R}$ *is s. t.* $\imath\xi \in \rho(\mathcal{A} + CN)$, *then* $\imath\xi \notin \sigma_p(\mathbb{A})$ *and* $\exists c(\xi) > 0$ *s. t.*

$$\|(\imath\xi\mathbb{I} - \mathbb{A})(U, P)^\top\|_{H\times X} \geq c(\xi)\|(U, P)^\top\|_{H\times X}, \forall (U, P)^\top \in D(\mathcal{A}) \times X,$$

in particular $R(\imath\xi\mathbb{I} - \mathbb{A})$ *is closed.*

Pf. Based on a contradiction argument.

Corollary (Coro 3)

Let (4) *and* (6) *be satisfied and suppose* $\exists$ *bd op.* $C : X \to H$ *s. t.* $\rho(\mathcal{A} + CN) \cap \sigma_p(-\mathbb{A}^\star) \cap \imath\mathbb{R} = \emptyset$. *Then*

$$\sigma(\mathbb{A}) \cap \imath\mathbb{R} \subset \sigma(\mathcal{A} + CN) \cap \imath\mathbb{R},$$

and if additionally $\sigma(\mathcal{A} + CN) \cap \imath\mathbb{R}$ *is countable, the semigroup* $T(t)$ *generated by* $\mathbb{A}$ *is stable.*

Frequency domain approach: exponential decay

Lemma (Prüss/Huang)

A C_0 semigroup $(e^{tA})_{t\geq 0}$ of contractions on a Hilbert space H is exponentially stable, i.e., satisfies

$$||e^{tA}U_0|| \leq C\,e^{-\omega t}||U_0||_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if and only if

$$\rho(A) \supset \left\{ i\beta \,\middle|\, \beta \in \mathbb{R} \right\} \equiv i\mathbb{R}, \tag{7}$$

$$\sup_{\beta\in\mathbb{R}} \|(i\beta - A)^{-1}\|_{\mathcal{L}(H)} < \infty. \tag{8}$$

Frequency domain approach: polynomial decay

Lemma (Borichev-Tomilov)

A C_0 semigroup $(e^{tA})_{t\geq 0}$ of contractions on a Hilbert space H satisfies

$$||e^{tA}U_0|| \leq C\, t^{-\frac{1}{\ell}}||U_0||_{\mathcal{D}(A)}, \quad \forall U_0 \in \mathcal{D}(A), \quad \forall t > 1,$$

for some constant $C > 0$ and for some positive integer ℓ if (7) holds and if

$$\limsup_{|\beta|\to\infty} \frac{1}{|\beta|^{\ell}} \|(i\beta - A)^{-1}\|_{\mathcal{L}(H)} < \infty. \tag{9}$$

A. Borichev and Y. Tomilov.
Optimal polynomial decay of functions and operator semigroups.
Math. Ann., 347(2):455–478, 2010.

The exponential case

Theorem

Assume that $\mathbb{A}$ (resp. $\mathcal{A}$) generates a bounded C_0 semigroup $T(t)$ (resp. $S(t)$) on $H \times X$ (resp. H) satisfying (7), namely $i\mathbb{R} \subset \rho(\mathbb{A})$ (resp. $i\mathbb{R} \subset \rho(\mathcal{A})$). Then
$T(t)$ is exponentially stable iff $S(t)$ is exponentially stable.

Pf. We show that

$$\|(\imath\xi\mathbb{I} - \mathcal{A})^{-1}\| \lesssim 1$$

iff

$$\|(\imath\xi\mathbb{I} - \mathbb{A})^{-1}\| \lesssim 1.$$

for $|\xi|$ large. Then we use Prüss/Huang's Theorem.

The polynomial case: one criterion

Theorem (Thm 4)

Assume that $\exists$ *a bd op.* $C : X \to H$ *s. t.* $\mathcal{A} + CN$ *generates a bounded* C_0 *semigroup on* H *satisfying* (7), *namely* $i\mathbb{R} \subset \rho(\mathcal{A} + CN)$, *and*

$$\sup_{\xi \in \mathbb{R}} \frac{1}{1 + |\xi|^m} \|(\imath\xi - (\mathcal{A} + CN))^{-1}\| < \infty, \tag{10}$$

for some non negative real number m. *Assume that* (4) *holds and that* $\mathbb{A}$ *generates a bounded* C_0 *semigroup* $T(t)$ *on* $H \times X$ *satisfying* (7), *namely* $i\mathbb{R} \subset \rho(\mathbb{A})$. *Then* $T(t)$ *is polynomially stable, i.e.,*

$$||T(t)(U_0, P_0)^\top||_{H \times X} \lesssim t^{-\frac{1}{\ell}} ||(U_0, P_0)^\top||_{D(\mathcal{A}) \times X}, \forall t > 1,$$

with $\ell = \max\{m, 2(m+1)\}$.

Use Borichev-Tomilov's Theorem and a contradiction argument.

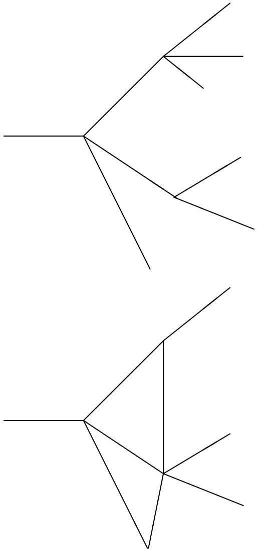

Figure: A tree shaped network and one with two cycles

The pb

A coupling between the telegraph equation and a first order ODE:

$$\begin{cases} V_{j,t} + g_j V_j + a_j I_{j,x} + k_j W_j = 0, & \text{in } Q_j := (0, l_j) \times (0, \infty), \forall j \in J, \\ I_{j,t} + r_j I_j + b_j V_{j,x} = 0, & \text{in } Q_j, \forall j \in J, \\ W_{j,t} + c_j W_j - V_j = 0, & \text{in } Q_j, \forall j \in J, \\ \sum_{j \in J_v} \nu_j(v) I_j(v,t) = 0, & \forall v \in V_{\text{int}}, t > 0, \\ V_j(v,t) - V_k(v,t) = 0, & \forall j,k \in J_v, \forall v \in V_{\text{int}}, t > 0, \\ V_{j_v}(v,t) = 0, & \forall v \in V_{\text{ext}}^{\text{Dir}}, t > 0, \\ V_{j_v}(v,t) - \alpha_v \nu_{j_v}(v) I_{j_v}(v,t) = 0, & \forall v \in V_{\text{ext}}^{\text{Diss}}, t > 0, \\ V(\cdot,0) = V_0, I(\cdot,0) = I_0, W(\cdot,0) = W_0 & \text{in } \mathcal{N}. \end{cases}$$

S. Imperiale and P. Joly. Mathematical modeling of electromagnetic wave propagation in heterogeneous lossy coaxial cables with variable cross section. *Appl. Numer. Math.*, 79:42-61, 2014.

The Hilbert setting

Unknowns: on each edge $e_j \equiv (0, \ell_j)$,

- the electric potential V_j,
- the electric current I_j,
- the non-local effects variable W_j.

boundary conditions: Kirchoff cdt on interior nodes, dissipative cdt on $\mathcal{V}_{\text{ext}}^{\text{Diss}}$.

Assumptions: a_j, b_j, c_j, k_j, r_j and g_j in $L^\infty(0, l_j)$ are real valued and non-negative functions satisfying

$$a_j \gtrsim 1, \; b_j \gtrsim 1, \; c_j \gtrsim 1, \; k_j + g_j \gtrsim 1 \quad \text{a.e. in } (0, l_j), \quad \forall j = 1, \dots, N.$$

These assumptions are in agreement with the physical setting from [Imperiale Joly' 14].

The Hilbert setting

Our system enters in the abstract framework (1) by defining H, X, $\mathcal{A}$, B, M and N as follows: $H = L^2(\mathcal{N})^2, X = L^2(\mathcal{N})$,

$$B : X \to X : W \to -cW,$$
$$M : X \to H : W \to (-kW, 0)^\top,$$
$$N : H \to X : (V, I)^\top \to V,$$

and are indeed bounded. Finally the operator $\mathcal{A}$ is defined as follows: the domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{(V, I)^\top \in PH^1(\mathcal{N})^2 \text{ satisfying the above bc}\},$$

$$\mathcal{A}(V, I)^\top = -(aI_x + gV, bV_x + rI)^\top, \forall (V, I)^\top \in D(\mathcal{A}).$$

With an appropriate choice of the inner products in H and X, $\mathbb{A}$ is dissipative, in particular (4) holds. Hence by Lemma 2, we obtain the next result.

The kernels on the imaginary axis

Lemma

One has

$$\ker(\imath\xi\mathbb{I}-\mathbb{A})=\{0\}, \forall\xi\in\mathbb{R}^*, \tag{11}$$

while

$$\ker\mathbb{A}=\{0\}\times K_0\times\{0\}, \tag{12}$$

where

$$K_0=\{l\in PP_0(\mathcal{N}) \textit{ satisfying the above bc and } rl=0\}.$$

Rk Different sufficient conditions on the network $\mathcal{N}$, the coefficient r and the choice of $V_{\rm ext}^{\rm Dir}$, $V_{\rm ext}^{\rm Diss}$ guarantee that $K_0=\{0\}$ (hence $\ker\mathbb{A}=\{0\}$). For instance, if $\mathcal{N}$ is a tree and $V_{\rm ext}^{\rm Diss}$ contains the set $V_{\rm ext}^*$ of all other exterior vertices except one, then $\ker\mathbb{A}=\{0\}$.

The polynomial stability result

Let us set

$$H_0 = L^2(\mathcal{N}) \times \{I \in L^2(\mathcal{N}) \mid \int_{\mathcal{N}} b^{-1} I \bar{I}' \, dx = 0, \forall I' \in K_0\},$$

then one can show that the restriction $\mathbb{A}_0$ of $\mathbb{A}$ to $H_0 \times X$ is well defined. Using Corollary 3 and Theorem 4 with C defined by

$$CW = -(kW, 0)^\top, \forall W \in X,$$

we get the next result.

Theorem

The semigroup $T_0(t)$ generated by $\mathbb{A}_0$ is polynomially stable, namely

$$\|T_0(t)(V, I, P)^\top\|_{H \times X} \lesssim t^{-\frac{1}{2}} \|(V, I, P)^\top\|_{D(\mathcal{A}) \times X}, \forall t > 1,$$

and all $(V, I, P)^\top \in (D(\mathcal{A}) \cap H_0) \times X$.


Thank you