Riemann–Hilbert problem on the torus: A case study

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Joint work with Gerald Teschl

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FWF Der Wissenschaftsfonds.

Aim of this talk

Goal: Case study of a R-H problem on a torus

R-H problem: Genus one KdV R-H problem (Its–Matveev solution)

Application: Deift–Zhou analysis of KdV solutions

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Finite gap KdV solutions

Theorem (Akhiezer, Dubrovin, Its, Matveev)

Algebro-geometric finite-gap solutions to the KdV equation with spectrum

$$[E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, \infty]$$

can be described explicitly in terms of the Jacobi theta function related to the hyperelliptic Riemann surface with upper sheet $\mathbb{C} \setminus \cup_{i=0}^{g} [E_{2i}, E_{2i+1}]$ *(*$E_{2g+1} = \infty$*) and related quantities:*

$$q(x,t) = 2\partial_x^2 \log \theta_3(Ux + Wt + D) + 2c$$

Belokolos, A. Bobenko, V. Enol'skii, A. Its and V. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer Series in Nonlinear Dynamics, Berlin, 1994.

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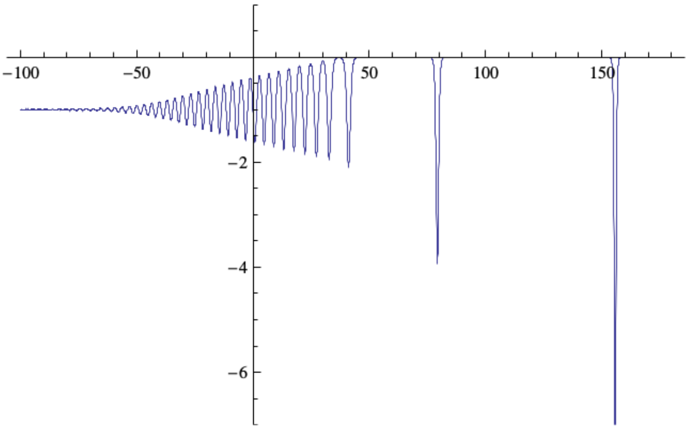

Figure: KdV solution with steplike initial data

Outline of the talk

1. KdV Riemann–Hilbert problem

2. Riemann–Hilbert problem on a torus

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Genus one KdV R-H problem

Find a vector-valued function, holomorphic in $\mathbb{C} \setminus [-\mathrm{i}c, \mathrm{i}c]$, $0 < a < c$,

$$m(k,x,t) = (m_1(k,x,t),\ m_2(k,x,t))$$

satisfying the **jump condition**,

$$m_+(k,x,t) = m_-(k,x,t)v(k,x,t), \quad k \in [-\mathrm{i}c, \mathrm{i}c]$$

$$v(k,x,t) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c], \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Omega} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\Omega} \end{pmatrix}, & k \in [\mathrm{i}a, -\mathrm{i}a], \end{cases}$$

with $\Omega(x,t) = Ux + Wt + D$.

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the **symmetry condition**,

$$m(-k,x,t) = m(k,x,t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

the **normalization condition**,

$$\lim_{k\to\infty} m(k,x,t) = (1,1).$$

and having at most **fourth root singularities** at $\pm \mathrm{i}a$:

$$m(k,x,t) = O((k \mp \mathrm{i}a)^{-1/4})$$

Question: How can we obtain a one gap solution $q(x,t)$ from $m(k,x,t)$?

Answer: $q(x,t) = \lim_{k\to\infty} 2k^2(m_1(k,x,t)m_2(k,x,t) - 1)$

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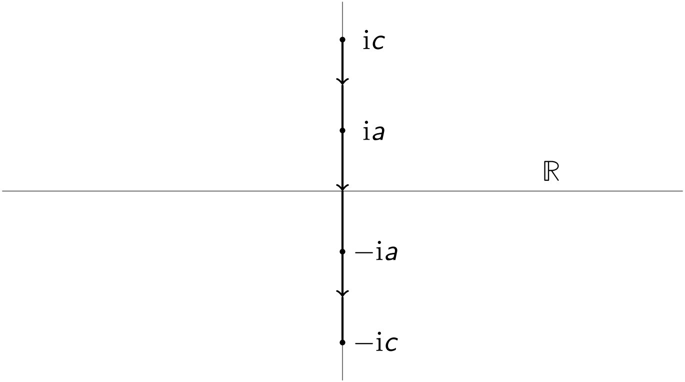

Figure: The jump contour

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1 KdV Riemann–Hilbert problem

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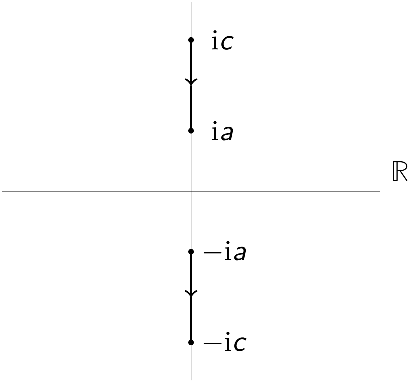
First step: scalar R-H problem

Find a scalar-valued function $\gamma\colon \mathbb{C}/([-\mathrm{i}c, -\mathrm{i}a] \cup [\mathrm{i}a, \mathrm{i}c]) \to \mathbb{C}$ s.t.:

- $\gamma_+(k) = \mathrm{i}\gamma_-(k), \quad k \in [\mathrm{i}a, \mathrm{i}c]$
- $\gamma_+(k) = -\mathrm{i}\gamma_-(k), \quad k \in [-\mathrm{i}a, -\mathrm{i}c]$
- $\lim_{k\to\infty} \gamma(k) = 1.$

$\Rightarrow$ Unique solution $\gamma(k) = \sqrt[4]{\frac{k^2+a^2}{k^2+c^2}}$



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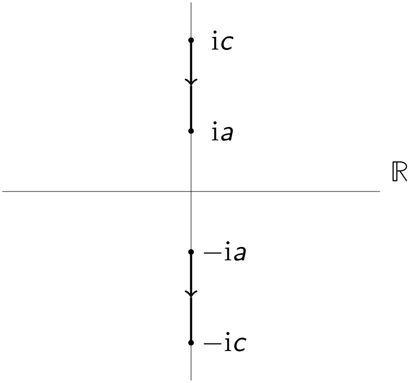

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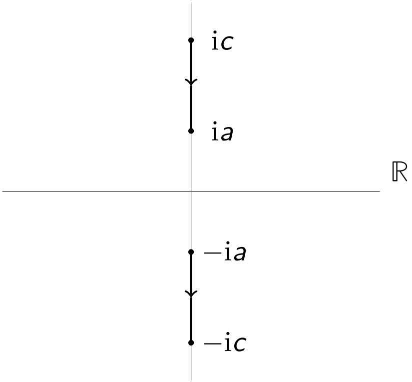

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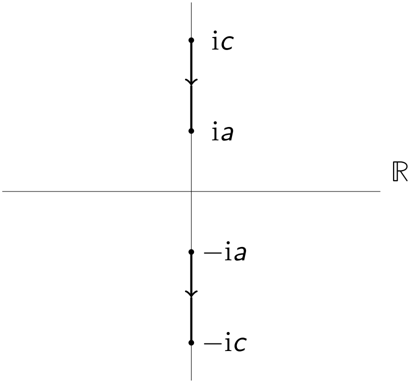

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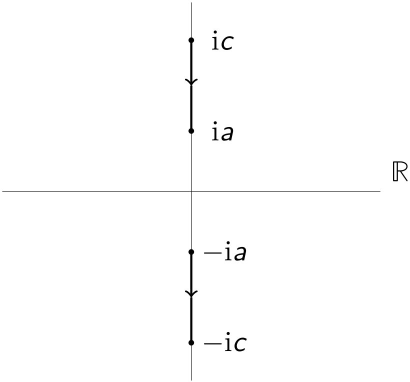

Second step: Conjugation

Define $n(k,x,t) := \gamma(k)^{-1} m(k,x,t)$, where $\gamma(k) = \sqrt[4]{\frac{k^2+a^2}{k^2+c^2}}$

Then

$$n_+(k,x,t) = n_-(k,x,t)\widetilde{v}(k,x,t), \quad k \in [-\mathrm{i}c, \mathrm{i}c]$$

$$\widetilde{v}(k,x,t) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c], \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Omega} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\Omega} \end{pmatrix}, & k \in [\mathrm{i}a, -\mathrm{i}a], \end{cases}$$

In particular $n_{1,\pm}(k,x,t) = n_{2,\mp}(k,x,t)$, for $k \in [-\mathrm{i}c, -\mathrm{i}a] \cup [\mathrm{i}a, \mathrm{i}c]$.

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Moreover, n satisfies the **symmetry condition**,

$$n(-k,x,t) = n(k,x,t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

the **normalization condition**,

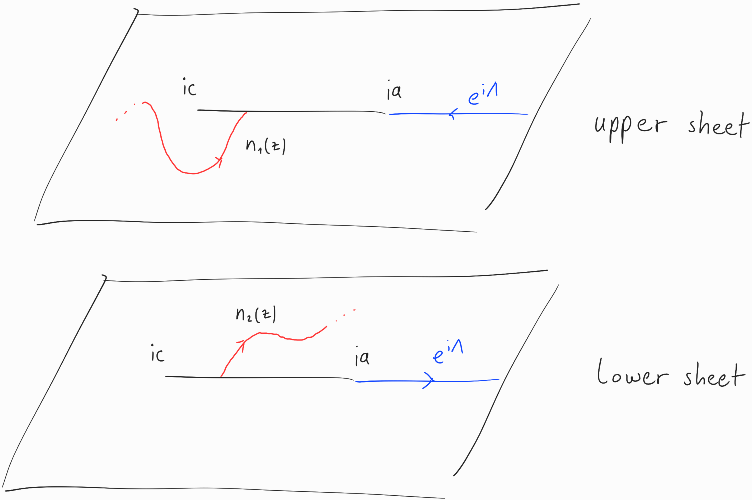
$$\lim_{k\to\infty} n(k,x,t) = (1,1).$$

and has at most **square root singularities** at $\pm \mathrm{i}a$:

$$n(k,x,t) = O((k \mp \mathrm{i}a)^{-1/2})$$

Second step: Conjugation

$$n_{1,\pm}(k,x,t) = n_{2,\mp}(k,x,t), \qquad k \in [-ic,-ia] \cup [ia, ic]$$



Second step: Conjugation

Define a function N on $\mathbf{X}$:

$$N((k,+)) = n_1(k), \quad \text{on the } \textbf{upper} \text{ sheet}$$
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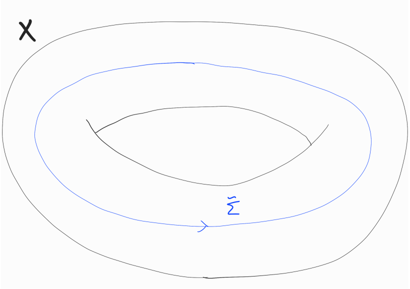
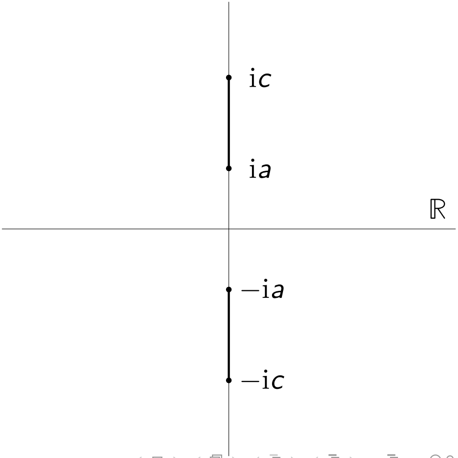


$$N_+(p) = N_-(p)\mathrm{e}^{\mathrm{i}\Omega}, \qquad p \in \widetilde{\Sigma}$$

Isomorphism between **X** and $\mathbb{C}/\Lambda$

We now construct an explicit bijection from **X** to the torus $\mathbb{C}/\Lambda$:



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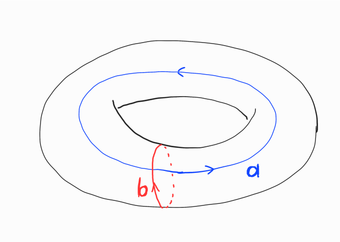
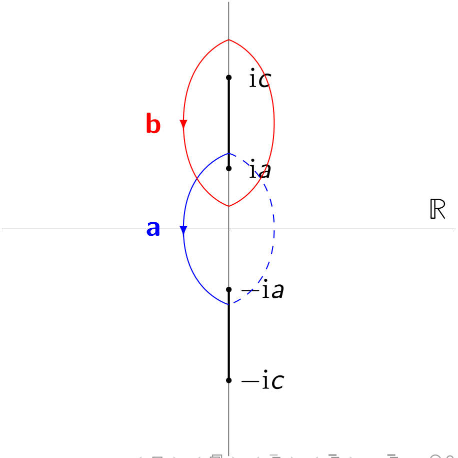
We define a homological basis **a**, **b**:

The cycles **a**, **b** describe fully the topology of **X**:





Isomorphism between X and $\mathbb{C}/\Lambda$

Next we define the normalized holomorphic differential $d\omega$ on **X**:

$$d\omega := \Gamma \frac{d\xi}{\sqrt{(\xi^2 + c^2)(\xi^2 + a^2)}}, \qquad \xi \in \mathbb{C} \setminus ([-\mathrm{i}c, \mathrm{i}a] \cup [\mathrm{i}a, \mathrm{i}c])$$

with $\Gamma := \left(\int_{\mathbf{a}} \frac{d\xi}{\sqrt{(\eta^2+c^2)(\eta^2+a^2)}} \right)^{-1} \Rightarrow \int_{\mathbf{a}} d\omega = 1.$

Define now (the half-period ratio)

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Isomorphism between $\mathbf{X}$ and $\mathbb{C}/\Lambda$

Having τ we can now define the **lattice**

$$\Lambda := \{m + n\tau \in \mathbb{C} | m, n \in \mathbb{Z}\}$$

and the **Abel map**

$$A : \mathbf{X} \to \mathbb{C}/\Lambda, \quad p \mapsto \int_{\mathrm{i}c}^{p} d\omega.$$

The Abel map A is an biholomorphism (holomorphic+invertible) between $\mathbf{X}$ and $\mathbb{C}/\Lambda$.

$\Rightarrow \mathbf{X} \simeq \mathbb{C}/\Lambda$ as Riemann surfaces.

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Dictionary between m and E

Define $E(A(p)) := N(p) \implies m(k) = (E(A(k)), E(-A(k)))$

Then $E(z+1) = E(z)$ and the following are equivalent:

$$m_+(k) = m_-(k) \begin{pmatrix} e^{-i\Omega} & 0 \\ 0 & e^{i\Omega} \end{pmatrix} \iff E(z+\tau) = E(z)e^{i\Omega}$$

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$$\lim_{k\to\infty} m(k) = (1,1) \iff E(\tfrac{1}{4}) = 1$$

fourth root singularities at $k = \pm ia \iff$ poles at $z = \dfrac{\tau}{2}, \dfrac{1+\tau}{2}$.

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fourth root singularities at $k = \pm \mathrm{i}a \iff$ poles at $z = \frac{\tau}{2}, \frac{1+\tau}{2}$.

Dictionary between m and E

Define $E(A(p)) := N(p) \implies m(k) = (E(A(k)), E(-A(k)))$

Then $E(z+1) = E(z)$ and the following are equivalent:

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Construction of E

Lemma

Let F be a meromorphic function satisfying

$$F(z+1) = F(z), \qquad F(z+\tau) = F(z)\mathrm{e}^{\mathrm{i}\Omega}$$

Then

$$F(z) = c \prod_{i=1}^{n} \frac{\theta_3(z - z_i - K)}{\theta_3(z - p_i - K)}$$

where $K = \frac{1+\tau}{2}$ and $\theta_3(z) = \sum_{n\in\mathbb{Z}} \exp\left((n^2\tau + 2nz)\pi\mathrm{i}\right)$ is the Jacobi theta function. The zeros z_i and poles p_i of F satisfy

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Pole condition $\Rightarrow p_1 = \frac{\tau}{2}, p_2 = \frac{1+\tau}{2}$

Symmetry condition $\Rightarrow z_2 = z_1 + \frac{1}{2}$

Quasiperiodicity (jump) condition $\Rightarrow z_1 + z_2 - p_1 - p_2 = \Omega$

$\Rightarrow E(z) = \frac{\theta_3(z - \frac{\Omega}{2} + \frac{1}{2})\theta_3(z - \frac{\Omega}{2})\theta_3(\frac{1}{4})^2}{\theta_3(z + \frac{1}{2})\theta_3(z)\theta_3(\frac{1}{4} - \frac{\Omega}{2})\theta_3(-\frac{1}{4} - \frac{\Omega}{2})}$, uniqueness for free

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Thank You!