On cohomogeneity one Hermitian non-Kähler manifolds

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Aim:

geometry of Hermitian metrics, invariant under the action of a compact Lie group G with principal orbits of codimension one

Why:

provide further examples of canonical Hermitian metrics

Riemannian geometry:

(Levi-Civita)

$\nabla g = 0, \; T = 0$

- constant sectional curvature
- Einstein
  homogeneous,
  Kähler-Einstein,
  cohomogeneity-one
- constant scalar curvature

Hermitian geometry:

(Chern)

$\nabla g = 0, \; \nabla J = 0, \; \nabla^{0,1} = \overline{\partial}$

- constant holomorphic sectional curvature
  $\Gamma\backslash G$ with G complex Lie group
- Chern-Einstein
- constant Chern-scalar curvature

Kähler

$$\nabla^{LC} = \nabla^{Ch}$$

Setting:

- $\underbrace{G}_{\text{cpt Lie grp}} \circlearrowleft \underbrace{(X, J, g)}_{\text{complete Herm}}$ effect by hol isom of cohomog-one

Structure:

- Orbit space $G \backslash X$ is homeomorphic to

$$\mathbb{R}, \qquad [0, +\infty), \qquad S^1, \qquad [0, 1]$$

  and there is a geodesic γ that intersects all the principal orbits
- Principal orbits are

$$G \cdot \gamma(r) \simeq G/H$$

- Non-principal orbits (if any) are

$$G/L_{\pm} \qquad \text{where } H \subseteq L_+ \cap L_- \subseteq G$$

  such that $L_{\pm}/H$ are spheres

Decompose $\mathfrak{g} = \underbrace{\mathfrak{h} + \overbrace{\mathfrak{a}}^{}}_{\mathfrak{k}}\!\!\overbrace{\phantom{\mathfrak{a}} + \mathfrak{p}}^{\mathfrak{m}}$

wrt $\mathrm{Ad}(G)$-inv $Q = -\frac{1}{2}\mathrm{tr}$

- $g|_{X^{\mathrm{reg}}} = dr^2 + \underbrace{g|_{G\cdot\gamma(r)}}_{\in \mathrm{Sym}^2(\mathfrak{m}^\vee)^{\mathrm{Ad}(H)}}$

  $\rightsquigarrow$ one-parameter family of metrics on G/H
- $T_r := {}^{J\gamma(r)}/_{|J\dot{\gamma}(r)|}$

  $\rightsquigarrow$ one-parameter family of CR-structures on G/H

Page metric:

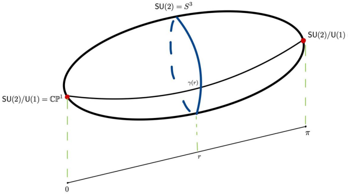
- Hopf fibration

$$\mathrm{U}(1) \to \mathrm{SU}(2) \to \mathbb{C}P^1$$

- associated projective bundle

$$\underbrace{\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbb{C}P^1}_{=\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}} \overset{\mathbb{C}P^1}{\to} \mathbb{C}P^1$$

- $\mathrm{SU}(2) \circlearrowleft \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$
  with principal orbits S^3 of codim one



(Picture courtesy of F. Pediconi.)

Page metric:

- $\mathrm{SU}(2)$-invariant metrics on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$
  $\rightsquigarrow$ (g_r) family of homogeneous metrics on $\mathrm{SU}(2)$
- *Assume:*
  g_r of submersion-type wrt $S^1 \to S^3 \to S^2$
  $\rightsquigarrow$ $f(r)$ =lenght of the fibre, $h(r)$ =scale factor of the base, with smoothness conditions smoothness (20)
- Einstein equation reduces to ODE

$$\begin{cases} \frac{h''}{h} - \frac{f'h'}{fh} + \frac{f^2}{h^4} = 0 \\ \frac{f''}{f} + \frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - \frac{2f^2}{h^4} = 0 \end{cases}$$

- $\rightsquigarrow$ $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ admits $\mathrm{SU}(2)$-inv Einstein metric with $s > 0$

Page metric:

- $\mathrm{SU}(2)$-invariant, Einstein metric with $s > 0$
- Hermitian, non-Kähler
- globally conformally Kähler
- not Kähler-Einstein, not Chern-Einstein
- $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ has Kähler metrics, but not Kähler-Einstein

Bérard-Bergery standard cohomogeneity-one manifolds:

- P Kähler-Einstein with $s > 0$,
  e.g. $P = G/K$ compact irreducible Hermitian symmetric space
- take the fibration

$$\mathrm{U}(1) \to \underbrace{{}^{G}/_{[K,K]} \times \mathbb{Z}_n}_{=:\Sigma_n} \to {}^{G}/_{K}$$

  classified by $c_1(\Sigma_n) = n\alpha$ where $c_1(P) = p\alpha$
- define
  - $X^1 := \Sigma_n \times \mathbb{R}$
  - $X^2 := \Sigma_n \times_{\mathrm{U}(1)} \mathbb{C} \overset{\text{diff}}{\simeq} \Sigma_n \times [0, +\infty)/\Sigma_n \times \{0\} \simeq P$
  - $X^3 := \Sigma_n \times S^1$
  - $X^2 := \Sigma_n \times_{\mathrm{U}(1)} \mathbb{C}P^1 \overset{\text{diff}}{\simeq} \Sigma_n \times [0, 1]/\Sigma_n \times \{0\} \simeq P \simeq \Sigma_n \times \{1\}$

  are holomorphic manifolds with $G \circlearrowleft X$ of cohomogeneity-one

	π_1	cpt	n.p.o.	examples
X^1			no	lens spaces
X^2	$\pi_1 = 1$		one	$\mathcal{O}_{\mathbb{C}P^{m-1}}(-n)$
X^3		cpt	no	diagonal Hopf
X^4	$\pi_1 = 1$	cpt	two	$\mathbb{C}P^m$ (21), $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^{m-1}} \oplus \mathcal{O}_{\mathbb{C}P^{m-1}}(-n))$ $\mathbb{C}P^m \sharp \overline{\mathbb{C}P^m}$ Hirzebruch surfaces

G-invariant Hermitian metrics of submersion-type are described by f and h satisfying smoothness conditions:

$$g = dr^2 + \frac{2m(m-1)n^2}{p^2} f(r)^2 Q|_{\mathfrak{a}\otimes\mathfrak{a}} + h(r)^2 Q|_{\mathfrak{p}\otimes\mathfrak{p}}$$

Theorem (Bérard-Bergery)

The ODEs system corresponding to the Einstein equation has solution on X^4.

Proposition (DA, Pediconi)

Let $X \in \mathfrak{p}$ with $Q(X,X) = 1$.

a) *The first Chern-Ricci tensor verifies*

$$\mathrm{Ric}^{\mathrm{Ch}[1]}(\mathfrak{g})(T^*,T^*)_{\mathfrak{g}(r)} = \frac{2m(m-1)n^2}{p^2} f(r)^2 \Big(-\frac{f''(r)}{f(r)} + (m-1)\Big(-\frac{h''(r)}{h(r)} + \frac{h'(r)^2}{h(r)^2} - \frac{f'(r)}{f(r)}\frac{h'(r)}{h(r)}\Big)\Big),$$

$$\mathrm{Ric}^{\mathrm{Ch}[1]}(\mathfrak{g})(X^*,X^*)_{\mathfrak{g}(r)} = h(r)^2 \Big(\frac{2mn}{p}\frac{f(r)}{h(r)^2}\Big(\frac{f'(r)}{f(r)} + (m-1)\frac{h'(r)}{h(r)}\Big) + \frac{2m}{h(r)^2}\Big).$$

b) *The second Chern-Ricci tensor verifies*

$$\mathrm{Ric}^{\mathrm{Ch}[2]}(\mathfrak{g})(T^*,T^*)_{\mathfrak{g}(r)} = \frac{2m(m-1)n^2}{p^2} f(r)^2 \Big(-\frac{f''(r)}{f(r)} + \frac{2m(m-1)n}{p}\frac{f'(r)}{h(r)^2} + \frac{2m^2(m-1)n^2}{p^2}\frac{f(r)^2}{h(r)^4}\Big),$$

$$\mathrm{Ric}^{\mathrm{Ch}[2]}(\mathfrak{g})(X^*,X^*)_{\mathfrak{g}(r)} = h(r)^2 \Big(-\frac{h''(r)}{h(r)} + \frac{h'(r)^2}{h(r)^2} - \frac{f'(r)}{f(r)}\frac{h'(r)}{h(r)} + \frac{2m(m-1)n}{p} f(r)\frac{h'(r)}{h(r)^3} - 2\Big(\frac{mn}{p}\Big)^2 \frac{f(r)^2}{h(r)^4} + \frac{2m}{h(r)^2}\Big).$$

d) *The Chern-scalar curvature is given by*

$$\mathrm{scal}^{\mathrm{Ch}}(\mathfrak{g})(r) = -2\frac{f''(r)}{f(r)} - 2(m-1)\frac{h''(r)}{h(r)} + 2(m-1)\Big(\frac{h'(r)}{h(r)} - \frac{f'(r)}{f(r)}\Big)\frac{h'(r)}{h(r)} + 4m(m-1)\frac{1}{h(r)^2} + \frac{4m(m-1)n}{p}\Big(f'(r) + (m-1)f(r)\frac{h'(r)}{h(r)}\Big)\frac{1}{h(r)^2}.$$

Second-Chern-Einstein equation:

$$\mathrm{Ric}^{\mathrm{Ch}[2]}(\mathbb{g}) = \frac{\lambda}{2m}\mathbb{g}$$

(Hermite-Einstein on T_X wrt itself)

$$\Leftrightarrow$$

$$\begin{cases} -\frac{f''(r)}{f(r)} + \frac{2m(m-1)n}{p}\frac{f'(r)}{h(r)^2} + \frac{2m^2(m-1)n^2}{p^2}\frac{f(r)^2}{h(r)^4} = \frac{\lambda(r)}{2m} \\ -\frac{h''(r)}{h(r)} + \frac{h'(r)^2}{h(r)^2} - \frac{f'(r)}{f(r)}\frac{h'(r)}{h(r)} + \frac{2m(m-1)n}{p}f(r)\frac{h'(r)}{h(r)^3} - 2\left(\frac{mn}{p}\right)^2\frac{f(r)^2}{h(r)^4} + \frac{2m}{h(r)^2} = \frac{\lambda(r)}{2m} \end{cases}$$

and smoothness conditions

Theorem (DA, Pediconi)

Local existence and uniqueness of second-Chern-Einstein metrics with prescribed Chern-scalar curvature in a neighbourhood of a singular orbit

(Malgrange theorem for initial value problem for ODEs)

Second-Chern-Einstein equation:

X^1, X^3 homogeneous Chern-Einstein

X^2 new complete Chern-Einstein metrics proof (23)

X^4 no cohom-one Chern-Einstein (no KE)

table (10)

Chern-Yamabe equation:

X^1, X^3 homogeneous Chern-Einstein

X^2 new complete Chern-Yamabe with $c \leq 0$ proof (24)

X^4 new Chern-Yamabe with $c > 0$

(including Hirzebruch surfaces [Koca-Lejmi])

table (10)

Locally conformally Kähler:

$d\omega = \theta \wedge \omega,\ d\theta = 0$

Theorem (DA, Pediconi)

Such metrics are always LCK or GCK.
They are Vaisman $(\nabla\theta = 0)$ *if and only if all orbits are principal and g is homogeneous*

Theorem (Gauduchon, Moroianu, Ornea; Hasegawa, Kamishima)

Homogeneous LCK metrics are Vaisman

Theorem (LeBrun; Derdziński, Maschler; Madani, Moroianu, Pilca)

Einstein LCK $\Rightarrow$ *Bérard-Bergery or* $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ *or* $\mathbb{C}P^2\sharp 2\overline{\mathbb{C}P^2}$

Theorem (DA, Pediconi)

- *The following are equivalent:*

  balanced ⇔ *SKT* ⇔ *Kähler*

  $d\omega^{n-1} = 0$ $\quad$ $\partial\bar{\partial}\omega = 0$ $\quad$ $d\omega = 0$

- *If there are non-principal orbits:*

  Gauduchon ⇔ *Kähler*

  $\partial\bar{\partial}\omega^{n-1} = 0$ $\quad$ $d\omega = 0$

Summary

- for homogeneous metrics: pde $\rightsquigarrow$ algebraic
- for cohomogeneity-one: pde $\rightsquigarrow$ ode
- examples coh-one include: $\mathbb{C}P^m$, Page, Hopf, Hirzebruch, ...
- $\rightsquigarrow$ new complete Chern-Einstein, new complete Chern-Yamabe
- $\rightsquigarrow$ characterize LCK, balanced, SKT, Gauduchon, ...

Grazie!

$$\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} = \mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbb{C}P^1 \stackrel{\mathbb{C}P^1}{\to} \mathbb{C}P^1$$

$\rightsquigarrow$ with $g = dr^2 + g_r$
$\rightsquigarrow$ with g_r of submersion-type wrt $S^1 \to S^3 \to S^2$
$\rightsquigarrow$ f and h such that

- $f\colon (0,\pi) \to \mathbb{R}^{>0}$ restriction of a smooth odd function on $\mathbb{R}$ such that $f(\pi + r) = -f(\pi - r)$, $f'(0) = 1 = -f'(\pi)$
- $h\colon (0,\pi) \to \mathbb{R}^{>0}$ restriction of a smooth even function on $\mathbb{R}$ such that $h(\pi + r) = h(\pi - r)$

back (7)

$\mathrm{SU}(m) \circlearrowleft \mathbb{C}P^m$

- $\mathfrak{g} = \mathfrak{su}(m) = \underbrace{\mathfrak{h} + \overbrace{\mathfrak{a}}^{}}_{\mathfrak{k}}\overbrace{{}+\mathfrak{p}}^{\mathfrak{m}}$ wrt $Q = -\frac{1}{2}\mathrm{tr}$

  where $\mathfrak{h} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array} \right) : A \in \mathfrak{su}(m-1) \right\}$

  $\mathfrak{a} = \left\{ \left(\begin{array}{c|c} -(m-1)t\sqrt{-1} & 0 \\ \hline 0 & t\sqrt{-1}\mathrm{Id}_{m-1} \end{array} \right) : t \in \mathbb{R} \right\}$

  $\mathfrak{p} = \left\{ \left(\begin{array}{c|c} 0 & X \\ \hline -\bar{X}^t & 0 \end{array} \right) : X \in \mathbb{C}^{m-1} \right\}$

- G-invariant metrics on $\mathbb{C}P^m$

  $\rightsquigarrow g = dr^2 + \underbrace{\frac{2(m-1)}{m} f(r)^2 Q_{\mathfrak{a}} + h(r)^2 Q_{\mathfrak{p}}}_{=:g_r \text{ invariant on } G/H = S^{2m-1}}$

  where $Q_{\mathfrak{a}}$ =standard metric on $K/H = S^1$ of radius 1

  $Q_{\mathfrak{p}}$ =Fubini-Study metric on $G/K = \mathbb{C}P^{m-1}$ of $1 \le \mathrm{scal} \le 4$

back (10)

$\mathrm{SU}(m) \circlearrowleft \mathbb{C}P^m$

- $f, h\colon (0, \frac{\pi}{2}) \to \mathbb{R}$ positive with smoothness conditions:
  - f is the restriction of a smooth odd function on $\mathbb{R}$ satisfying $f(r + \frac{\pi}{2}) = -f(r - \frac{\pi}{2})$ and $f'(0) = 1 = -f'(\frac{\pi}{2})$
  - h is the restriction of a smooth even function on $\mathbb{R}$ satisfying $h(r + \frac{\pi}{2}) = -h(r - \frac{\pi}{2})$ and $h'(\frac{\pi}{2}) = -1$
- Fubini-Study: $f(r) = \frac{1}{2}\sin(2r)$, $h(r) = \cos r$

back (10)

Theorem (DA, Pediconi)

X^2 admit complete, non-Kähler, cohomogeneity-one, second-Chern-Einstein metrics, with non-constant Chern-scalar curvature back (14)

Proof.

- $f_\phi(r) = \frac{p}{2mn}\phi(r)\phi'(r)$ and $h_\phi(r) = \phi(r)$ for ϕ smooth, positive, increasing

$$\rightsquigarrow \phi(r)^2 \frac{\phi'''(r)}{\phi'(r)} + (m+2)\phi(r)\phi''(r) - m(m-1)\phi'(r)^2 + \frac{c}{2}\phi(r)^2 - 2m(m-1) = 0$$

- $\phi'(r) = \sqrt{u(\phi(r))}$

$$\rightsquigarrow t^2 u''(t) + (m+2)tu'(t) - 2m(m-1)u(t) + ct^2 - 4m(m-1) = 0$$

- $u_{a,b,c}(t) = at^{-2m} - 2 + \frac{c}{2(m+1)(m-3)}t^2 + bt^{m-1}$
- Smoothness conditions $\phi(0) = 1$, $\phi'(0) = 0$, $\phi''(0) = \frac{2mn}{p}$ determine

$$a(c) = -\frac{(m+1)(4m(2n-p)+4p)+cp}{2p(m+1)(3m-1)}, \qquad b(c) = \frac{4m(m-3)(n+p)-cp}{p(3m-1)(m-3)}$$

$\square$

Theorem (DA, Pediconi)

X^2 admit complete, non-Kähler, cohomogeneity-one metrics with non-positive constant Chern-scalar curvature back (14)

Proof.

- $f_\phi(r) = \frac{p}{2mn}\phi'(r)$ and $h_\phi(r) = \sqrt{\phi(r)}$ for ϕ smooth, positive, increasing

$$\rightsquigarrow \begin{cases} \phi(r)\frac{\phi'''(r)}{\phi'(r)} - m\phi''(r) + 2m = 0 \\ \phi(0) = 1, \quad \phi'(0) = 0, \quad \phi''(0) = \frac{2mn}{p} \end{cases}$$

- $\phi'(r) = \sqrt{u(\phi(r))}$

$$\rightsquigarrow \begin{cases} tu''(t) - mu'(t) + 4m = 0 \\ u(1) = 0, \quad u'(1) = \frac{4mn}{p} \end{cases}$$

- $u(t) = -\frac{4m(n+p)}{p(m+1)} + 4t + \frac{4(mn-p)}{p(m+1)}t^{m+1}$, $\phi(r) = \int_1^r \frac{dt}{\sqrt{u(t)}}$

□