Optimal Drawdowns in Insurance

joint work with Leonie Brinker

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1. Drawdowns
   - Introduction
   - The Classical Risk Model
   - The Diffusion Approximation

Definition of Drawdown

For a surplus process X_t denote by

$$\bar{X}_t = \max\{\bar{x}, \sup_{s \le t} X_s\}$$

the running maximum. The drawdown

$$D_t = \bar{X}_t - X_t$$

is the deviation from the running maximum. We allow a past maximum $\bar{x}$.

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- Goal is to keep surplus near maximum (stabilsation) which simplifies planning future strategies
- We try to keep the drawdown below some level d
- Drawdown below the critical level only for a short time

Reinsurance

The insurer buys proportional reinsurance with retention level $b_t \in [0,1]$ at time t. That is, the insurer pays $b_t Y$, the reinsurer $(1-b_t)Y$ of a claim of size Y. The reinsurer uses an expected value principle with safty loading θ. We assume that reinsurance is more expensive than first insurance in order that the problem below is not trivial. The insurer choses continuously a reinsurance strategy $\{b_t\}$.

The Optimisation Problem

The value of a reinsurance strategy b is

$$V^b(x) = \mathbb{E}\Bigl[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{D_t^b > d} \,\mathrm{d}t\Bigr] .$$

We are interested in the optimal value

$$V(x) = \inf_b V^b(x)$$

and, if it exist, the optimal strategy b^*.

The Cramér–Lundberg Model

Let

$$X_t = \bar{x} - x + ct - \sum_{k=1}^{N_t} Y_k \ ,$$

where N is a Poisson process with rate λ and iid claim $\{Y_k\}$ with expected value μ. We write $c = (1+\eta)\lambda\mu$ for some $\eta > 0$.

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After reinsurance,

$$X_t^b = \bar{x} - x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{k=1}^{N_t} b_{T_k-} Y_k \ ,$$

where $c(b) = c - (1-b)(1+\theta)\lambda\mu = (b\theta - (\theta - \eta))\lambda\mu$.

The Drawdown Process

We get the drawdown process

$$D_t^b = x + \sum_{k=1}^{N_t} b_{T_k -} Y_k - \int_0^t c(b_s)\,\mathrm{d}s + (\bar{X}_t^b - \bar{x})\;.$$

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- jumps upwards, (downwards) deterministic paths
- reflection in zero
- we now restrict to $b_t \in [b^0, 1]$ with $b^0 = (1 - \eta/\theta)$, such that $c(b_t) \geq 0$.

The Diffusion Approximation

With simplified notation, the diffusion approximation to the classical model is $X_t = \bar{x} - x + \eta t + \sigma W_t$ for some Brownian motion W. After reinsurance

$$X_t^b = \bar{x} - x + \int_0^t \{b_s\theta - (\theta - \eta)\}\,\mathrm{d}s + \sigma \int_0^t b_s\,\mathrm{d}W_s\,.$$

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The drawdown process becomes

$$D_t^b = x - \int_0^t \{b_s\theta - (\theta - \eta)\} \,\mathrm{d}s - \sigma \int_0^t b_s \,\mathrm{d}W_s + (\bar{X}_t^b - \bar{x}) \,.$$

Lipschitz Continuity

Lemma

The function V is increasing with $0 \le V(x) \le \delta^{-1}$ for all $x \in [0, \infty)$, fulfils $\lim_{x \to \infty} V(x) = \delta^{-1}$ and is Lipschitz continuous with

$$|V(x) - V(y)| \le \frac{\lambda + \delta}{\delta c(1)} |x - y| \; .$$

In particular, V is absolutely continuous and differentiable almost everywhere.

Proof.

For $0 \le y < x$ choose a strategy $\tilde{b}$ with $V^{\tilde{b}}(y) < V(y) + \varepsilon$. For initial capital x define $h = (x-y)/c(1)$, $b_t = \tilde{b}_{t-h}$ if $T_1 \wedge t \ge h$ and $b_t = 1$, otherwise. Then

$$\begin{aligned} V(x) - V(y) - \varepsilon \le V^b(x) - V^{\tilde{b}}(y) \\ \le \int_0^h \mathrm{e}^{-\delta t}\,\mathrm{d}t - (1 - \mathrm{e}^{-(\lambda+\delta)h})V^{\tilde{b}}(y) + (1 - \mathrm{e}^{-\lambda h})\delta^{-1} \\ \le (\lambda+\delta)h/\delta = \frac{\lambda+\delta}{\delta c(1)}(x-y)\,, \end{aligned}$$

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The other statements are clear. □

Splitting of the Problem

Let

$$\vartheta_d = \inf\{t \ge 0 : D_t \le d\}\;, \qquad \vartheta^d = \inf\{t \ge 0 : D_t > d\}$$

be the first entrance times. Then by considering the process until the stopping time

$$\begin{aligned} V(x) &= \mathbb{E}[e^{-\delta\vartheta^d} V(D_{\vartheta^d})]\;, & x \le d\;, \\ V(x) &= \mathbb{E}[\delta^{-1}(1 - e^{-\delta\vartheta_d}) + e^{-\delta\vartheta_d} V(d)]\;, & x > d\;. \end{aligned}$$

We can solve the two problems separately.

Starting in the Critical Area

Problem: Maximise $\mathbb{E}^x[\mathrm{e}^{-\delta\vartheta_d}]$.

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$V(x) = \delta^{-1} - (\delta^{-1} - V(d))\mathrm{e}^{-\gamma(x-d)}$ where γ is the positive solution to $c(1)\gamma - \lambda\mathbb{E}[1 - \mathrm{e}^{-\gamma Y}] = \delta$.

Starting in the Non-Critical Area

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Replace $V(d)$ by $C \in (0, \delta^{-1})$, $V_C(x) = \inf_b \mathbb{E}[\mathrm{e}^{-\delta\vartheta^d} V_C(D_{\vartheta^d})]$.

Lemma

There exists $C_0 \in (0, \delta^{-1})$ such that $V_C(d) \gtreqless C$ iff $C \lesseqgtr C_0$.

It turns out that $C_0 = V(d)$.

The HJB Equation

Theorem

$V_C(x)$ solves for $x \le d$ the HJB equation

$$\inf_{b\in[b^0,1]} \lambda \int_0^\infty V_C(x+by)\,\mathrm{d}G(y) - c(b)V_C'(x) - (\lambda+\delta)V_C(x) = 0\;.$$

Let $b_C(x)$ be a measurable version of the maximiser. Then the strategy $b_C(D_t^C)$ is optimal.

The HJB Equation II

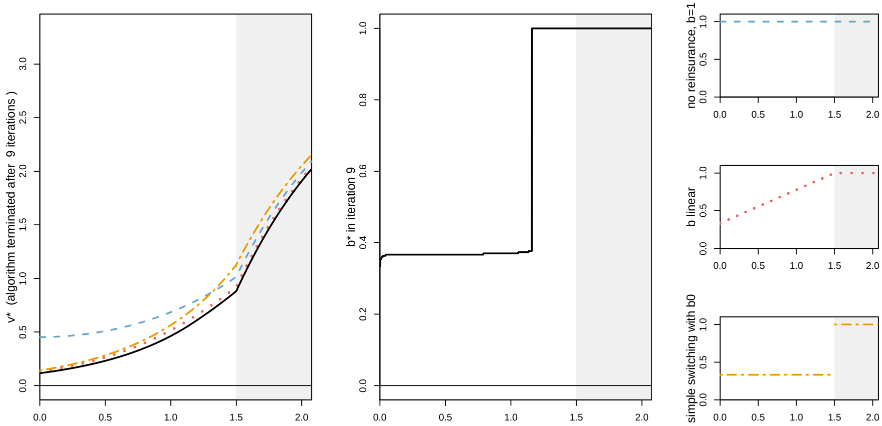
Theorem

$V(x)$ is the unique bounded continuous solution to the HJB equation

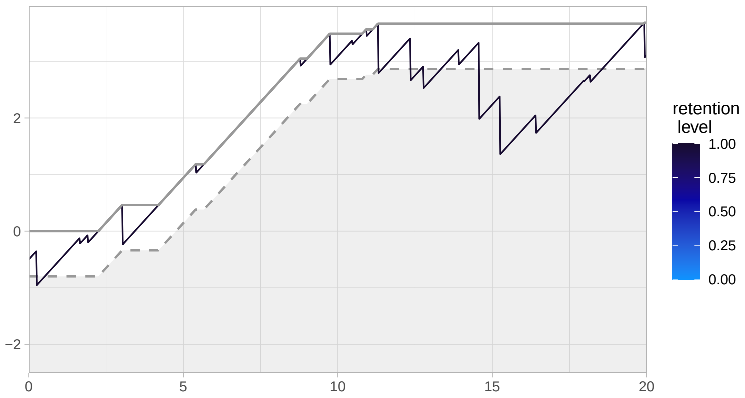
$$\inf_{b\in[b^0,1]} \lambda\int_0^\infty V(x+by)\,\mathrm{d}G(y)-c(b)V'(x)-(\lambda+\delta)V(x) = -\mathbb{1}_{x>d}\;.$$

Let $b(x)$ be a measurable version of the maximiser. Then the strategy $b(D^*_t)$ is optimal.

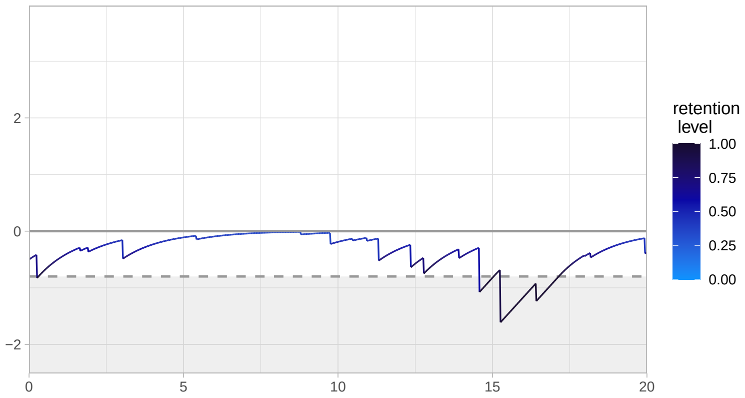
Exponentially Distributed Claims



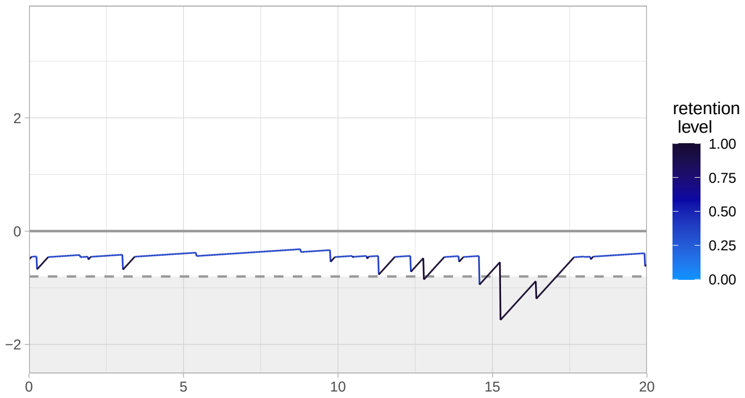
Exponentially Distributed Claims: No Reinsurance



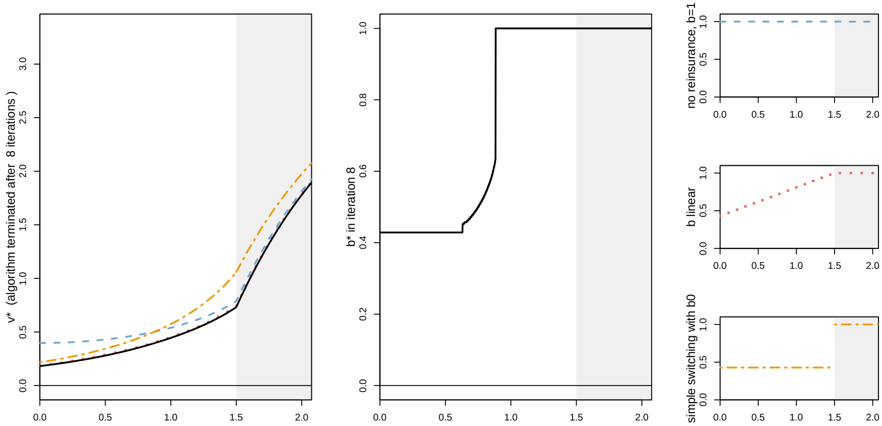
Exponentially Distributed Claims: Linear Reinsurance



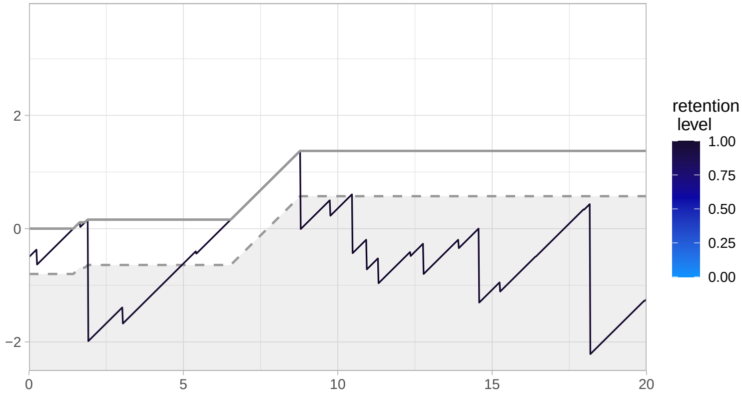
Exponentially Distributed Claims: Optimal Reinsurance



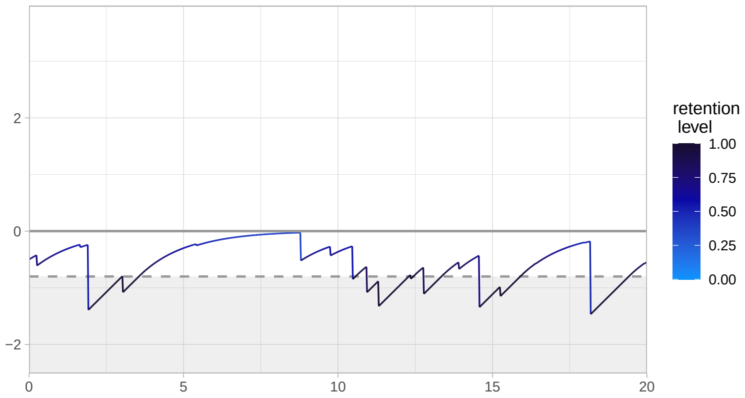
Pareto Distributed Claims



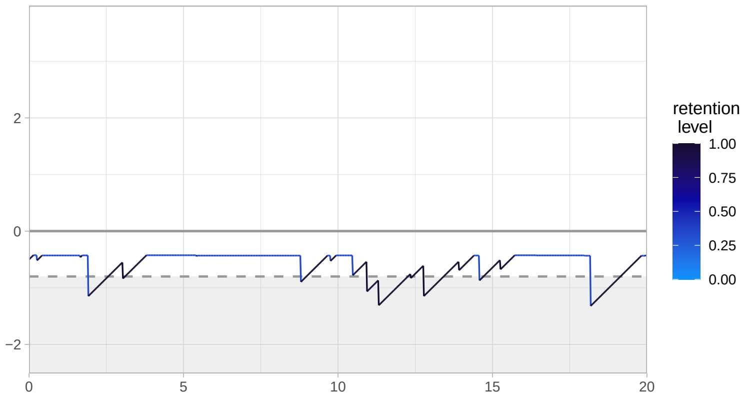
Pareto Distributed Claims: No Reinsurance



Pareto Distributed Claims: Linear Reinsurance



Pareto Distributed Claims: Optimal Reinsurance



Splitting of the Problem

As for the classical model

$$\begin{aligned} V(x) &= \mathbb{E}[\mathrm{e}^{-\delta\vartheta^d} V(d)]\,, & x \le d\,, \\ V(x) &= \mathbb{E}[\delta^{-1}(1-\mathrm{e}^{-\delta\vartheta_d}) + \mathrm{e}^{-\delta\vartheta_d} V(d)]\,, & x > d\,. \end{aligned}$$

In the critical area $x > d$ $b = 1$ and thus

$$V(x) = \delta^{-1}\{1 - (1-\delta V(d))\mathrm{e}^{-\kappa(x-d)}\}$$

for $\kappa > 0$ solving $\frac{1}{2}\sigma^2\kappa^2 + \eta\kappa = \delta$.

The HJB Equation

Theorem

$V(x)$ is the unique bounded continuously differentiable solution to

$$(\theta-\eta)V'(x)-\delta V(x)+\inf_{b\in[0,1]}\Big\{\frac{1}{2}b^2\sigma^2V''(x)-\theta bV'(x)\Big\}=-\mathbb{1}_{x>d}\;.$$

Proof.

Explicit solution to the HJB and verification theorem. □

Construction of the Solution

A non-trivial solution must be strictly convex. If $b \neq 1$,

$$\frac{\theta^2 V'(x)^2}{2\sigma^2 V''(x)} + \delta V(x) = (\theta - \eta) V'(x) \ .$$

The function $x \mapsto -\ln V'(x)$ is strictly decreasing with inverse function Y. Thus $V'(Y(z)) = \mathrm{e}^{-z}$. Plugging this into the equation and differentiation leads to differential equation and an explicit solution. There is $x_0 \in (0, \infty]$ such that

$$b(x) = \frac{\theta V'(x)}{\sigma^2 V''(x)} \le 1 \ , \quad x \in [0, x_0] \ .$$

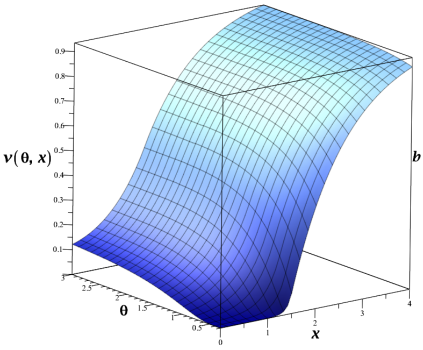
Compound $V(x)$ on $[0, x_0 \wedge d]$ with the solution with $b(x) = 1$ to a smooth solution.

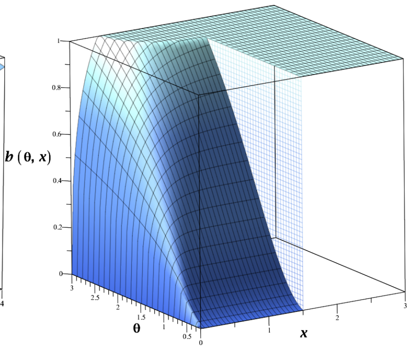
The Behaviour at Zero

Theorem

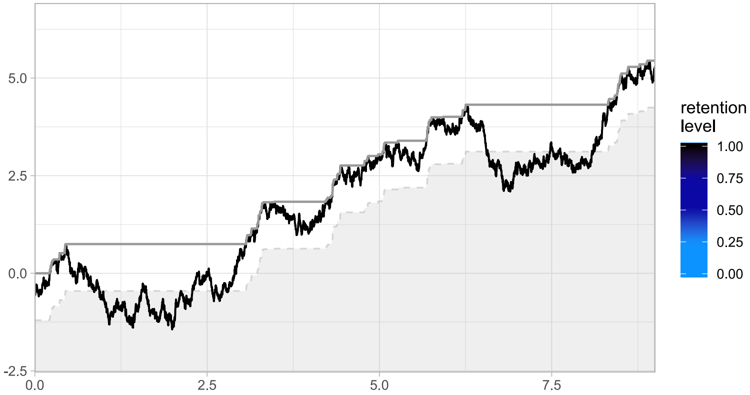
The strategy $b(D_t^*)$ is optimal. Under the optimal strategy $\bar{X}_t^*$ is constant.

Value Function and $b(x)$

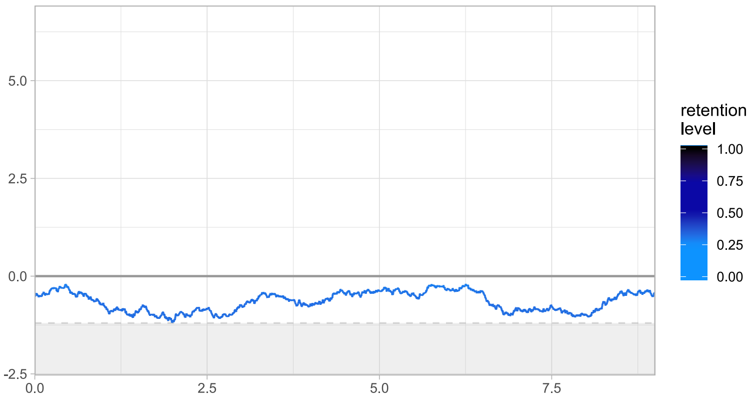




No Reinsurance



Optimal Reinsurance



References

- Brinker, L. and S. (2021). Optimal discounted drawdowns in a diffusion approximation under proportional reinsurance. *Preprint.*
- Brinker, L. and S. (2021). Optimal drawdowns in the classical risk model under proportional reinsurance. *Preprint.*
- S. (2008). *Stochastic Control in Insurance*. Springer-Verlag, London.

Thank you for your attention