Stochastic completeness and uniqueness class for graphs

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Outline

Heat equation on $\mathbb{R}$

Consider the Cauchy problem for the heat equation:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) + \Delta u(x,t) = 0, \\ u(\cdot, 0) \equiv 0; \end{cases} \qquad (\spadesuit)$$

Here $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, with $\Delta u(x,t) = -\frac{\partial^2}{\partial x^2} u(x,t)$.

Tichonov solution

Natural solution: $u \equiv 0$;
Tichonov solution:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^k(t)}{(2k)!} x^{2k},$$

where

$$g(t) = \begin{cases} \exp(t^{-2}), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

$|u(x,t)|$ can be bounded at best by $\exp\big(C(\varepsilon)\,|x|^{2+\varepsilon}\big)$.

Täcklind's uniqueness class

Täcklind proved that if $|u(x,t)| \leq h(|x|)$ for $|x|$ large, where

$$\int^{\infty} \frac{r}{\ln h(r)} \,\mathrm{d}r = +\infty,$$

then $u \equiv 0$. The solution to the Cauchy problem (♠) is unique in such a class of functions.
In particular, bounded functions form a uniqueness class.

Stochastic completeness

The Laplacian Δ generates a semigroup of operators

$$P_t = \exp(-t\Delta), t \geq 0.$$

It is closely related to the Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}$:

$$P_t f(x) = \mathbb{E}_x\big(f(B_t)\big).$$

Bounded solutions form a uniqueness class $\iff$ stochastic completeness, that is,

$$P_t \mathbf{1} = \mathbf{1}.$$

(Note that $\mathbf{1} - P_t\mathbf{1}$ is a bounded solution to the Cauchy problem.)

Heat equation on manifolds

Let (M, g) be a complete Riemannian manifold with the Laplace-Beltrami operator Δ (≥ 0).
Consider the Cauchy problem for the heat equation:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) + \Delta u(x,t) = 0, \\ u(\cdot, 0) \equiv 0. \end{cases} \qquad (\spadesuit)$$

Grigor'yan's uniqueness class

Theorem (Grigor'yan)

If $u : M \times [0, T] \to \mathbb{R}$ *solves* (♠) *and satisfies*

$$\int_0^T \int_{B(\bar{x},r)} u^2(x,t)\, \mathrm{d}\,\mathrm{vol}(x)\, \mathrm{d}t \leq h(r)$$

for r *large, where*

$$\int^{\infty} \frac{r}{\ln h(r)}\, \mathrm{d}r = +\infty,$$

then $u \equiv 0$.

Proof strategy

A localized version of monotonicity formula:

$$\frac{d}{dt}\int_M u^2(x,t)\exp\xi(x,t)\,\mathrm{d}\,\mathrm{vol}(x)\leq 0,$$

where ξ satisfies

$$\frac{\partial}{\partial t}\xi(x,t)+\frac{1}{2}\left|\nabla\xi(x,t)\right|^2\leq 0.$$

For example: $\xi(x,t)=-\frac{d(\bar{x},x)^2}{2t}$.

Stochastic completeness

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Volume growth criteria for stochastic completeness

The uniqueness class theorem, when applied to bounded solutions, implies a sharp volume growth type criterion for stochastic completeness.

Theorem (Grigor'yan)

Suppose

$$\int^{\infty} \frac{rdr}{\ln\big(\mathrm{vol}\big(B_d(\bar{x}, r)\big)\big)} = \infty, \qquad (\dagger)$$

then the Brownian motion on (M, g) *is stochastically complete.*

Heat equation on $\mathbb{Z}$

What happens for graphs?

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) + \Delta u(x,t) = 0, \\ u(\cdot, 0) \equiv 0. \end{cases} \qquad (\spadesuit)$$

Here $u : \mathbb{Z} \times [0, T] \to \mathbb{R}$, with

$$\Delta u(n,t) = 2u(n,t) - \big(u(n-1,t) + u(n+1,t)\big).$$

Tichonov type solution

Natural solution: $u \equiv 0$;
Tichonov type solution:

$$u(n,t) = \begin{cases} g(t), & n = 0; \\ \sum_{k=0}^{\infty} \frac{g^k(t)}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1), & n \geq 1; \\ u(-n-1,t), & n \leq -1. \end{cases}$$

where

$$g(t) = \begin{cases} \exp(t^{-2}), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Growth

Note that for $n \geq 1$,

$$\sum_{k=0}^{\infty} \frac{g^k(t)}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1) = \sum_{k=0}^{n} \cdots .$$

In contrary to the smooth case, for large $|n|$,

$$|u(n,t)| \leq \exp\big(C\,|n|\ln|n|\big).$$

Questions

- What about the uniqueness class for general weighted graphs? We cannot expect growth conditions as large as the smooth case.
- What is the sharp volume growth type criterion for stochastic completeness of weighted graphs?

Weighted graphs

Let V be a discrete countable set with weights:

- $\mu : V \to (0, \infty)$, as a measure on V;
- $w : V \times V \to [0, \infty)$
  - a $w(x, y) = w(y, x)$;
  - b $w(x, x) = 0$;
  - c $\sum_{y \in V} w(x, y) < +\infty$.

Denote $x \sim y$ when $w(x, y) > 0$. We assume connectedness.
The formal Laplacian:

$$(\Delta f)(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(f(x) - f(y)).$$

The heat semigroup

The Laplacian Δ generates the heat semigroup

$$P_t = \exp(-t\Delta), t \geq 0,$$

which corresponds to a minimal continuous time Markov chain on V.
Bounded solutions form a uniqueness class for the Cauchy problem (♠) of the heat equation $\iff$ stochastic completeness, that is,

$$P_t\mathbf{1} = \mathbf{1}.$$

$\mathbb{Z}$ with weights

$V = \mathbb{Z}$, with $n \sim n+1$, as a graph.

- $\mu(n) \equiv 1$, $w(n, n+1) \equiv 1$;
- $\mu(n) \equiv 1$, $w(0, -1) = 1$,
  $w(n-1, n) = w(-n, -n-1) = n$ for $n \geq 1$.

Intrinsic metrics

Definition

A metric d on (V, w, μ) is called an intrinsic metric if

$$\forall x \in V, \qquad \frac{1}{\mu(x)} \sum_{y \in V} w(x,y) d(x,y)^2 \leq 1. \qquad (\diamondsuit)$$

Remark

An intrinsic metric is sensible to the weights μ, w. Condition $(\diamondsuit)$ is a discrete analogue of $|\nabla d(\bar{x}, \cdot)| \leq 1$. For simplicity, we also assume bounded jump size: $d(x,y) \leq \sigma_0$ whenever $x \sim y$.

Examples of intrinsic metrics

$V = \mathbb{Z}$, with $n \sim n+1$, as a graph.

- $\mu(n) \equiv 1$, $w(n, n+1) \equiv 1$; let $d(n, n+1) \equiv \frac{\sqrt{2}}{2}$ which is naturally extended to a shortest path metric.
- $\mu(n) \equiv 1$, $w(0, -1) = 1$,
  $w(n-1, n) = w(-n, -n-1) = n$ for $n \geq 1$; let

$$d(n-1, n) = \sqrt{\frac{1}{2 \vee (2|n|+1)}},$$

which is naturally extended to a shortest path metric.

Uniqueness class

Theorem (H.)

Under some mild conditions, for some constant $c > 0$, if $u : V \times [0, T] \to \mathbb{R}$ solves the Cauchy problem (♠) and satisfies

$$\int_0^T \int_{B(\bar{x},r)} u^2(x,t)\,\mathrm{d}\mu(x)\,\mathrm{d}t \le \exp\big(c\sigma_0 r \ln r\big)$$

for r large, then $u \equiv 0$.

Remark

As a consequence, if $\mu\big(B(\bar{x}, r)\big) \le \exp\big(c\sigma_0 r \ln r\big)$ for r large, then the corresponding Markov chain is stochastically complete.

Difficulties

Lack of chain rule: unlike

$$\left|\nabla \exp \xi(x)\right| \leq \exp \xi(x) \left|\nabla \xi(x)\right|,$$

we have at best

$$\frac{1}{\mu(x)} \sum_{y \in V} w(x,y) \big(\exp \xi(x) - \exp \xi(y) \big)^2$$
$$\leq \exp 2 \big(\xi(x) \vee \xi(y) \big) \frac{1}{\mu(x)} \sum_{y \in V} w(x,y) \big(\xi(x) - \xi(y) \big)^2.$$

Stochastic completeness

Theorem (Folz)

Under some technical conditions, if

$$\int^{\infty} \frac{r \, \mathrm{d}r}{\ln \mu\big(B(\bar{x}, r)\big)} = \infty,$$

then (V, w, μ) *is stochastically complete.*

Remark

Folz works by relating the Markov chain to a diffusion. Stochastic completeness is about "very large" time property, and is much more stable than the uniqueness class is (which involves short time information as well).

Goals of the present work

- to recover Grigor'yan's uniqueness class for a certain special class of weighted graphs;
- to apply stability arguments to obtain a generalized version of Folz's volume growth criterion.

GL (globally local) condition

Let

$$s_r := \sup\{d(x,y) \mid x,y \in X \text{ with } x \sim y \text{ and } d(x,\bar{x}) \wedge d(y,\bar{x}) \geq r\}.$$

Definition

A weighted graph (V,w,μ) with an intrinsic metric d is called globally local with respect to an increasing function $f\colon (0,\infty) \to (0,\infty)$ if there is a constant $A > 1$ such that

$$\limsup_{r\to\infty} \frac{s_r f(Ar)}{r} < \infty. \tag{GL}$$

Uniqueness class under the GL condition

Theorem (H., Keller, Schmidt)

Let a weighted graph (V, w, μ) with an intrinsic metric d be globally local with respect to an increasing function $f\colon (0,\infty) \to (0,\infty)$ with $\int^{\infty} \frac{r}{f(r)}\,\mathrm{d}r = +\infty$. Assume that balls in d are finite. If $u : V \times [0, T] \to \mathbb{R}$ solves the Cauchy problem (♠) and satisfies

$$\int_0^T \int_{B(\bar{x},r)} u^2(x,t)\,\mathrm{d}\mu(x)\,\mathrm{d}t \leq \exp f(r)$$

for r large, then $u \equiv 0$.

Stochastic completeness

Theorem (H., Keller, Schmidt)

Let (V, w, μ) be a weighted graph with an intrinsic metric d such that balls in d are finite. If

$$\int^{\infty} \frac{r\,\mathrm{d}r}{\ln \mu\big(B(\bar{x}, r)\big)} = \infty,$$

then (V, w, μ) is stochastically complete.

Stability and modifications of weighted graphs

Main ingredients:

- a “piecing out” argument to deal with unbounded jump size;
- adding new vertices to the original weighted graph to split big jumps into smaller steps (a globally local one);
- a potential theoretic argument (the weak Omori-Yau maximum principle) for stability of stochastic completeness under modifications.

A sharpness example

$V = \mathbb{Z}$, with $n \sim n+1$, as a graph. Given weights $\mu(n) \equiv 1$, $w(0,-1) = 1$, $w(n-1,n) = w(-n,-n-1) = n$ for $n \geq 1$. Let

$$d(n-1,n) = \sqrt{\frac{1}{2 \vee (2\,|n| + 1)}},$$

which is naturally extended to a shortest path metric.

A sharpness example

We have $d(0,n) \simeq \sqrt{|n|}$, and $s_r \simeq \frac{1}{r}$ for r large.
A Tichonov type solution:

$$u(n,t) = \begin{cases} g(t), & n = 0; \\ \displaystyle\sum_{k=0}^{\infty} \binom{n}{k} \frac{g^k(t)}{k!}, & n \geq 1; \\ u(-n-1,t), & n \leq -1. \end{cases}$$

A sharpness example

Bound:

$$\int_0^T \int_{B(\bar{x},r)} u^2(x,t)\,\mathrm{d}\mu(x)\,\mathrm{d}t \le \exp\big(Cr^2 \ln r\big)$$

for r large.
Note

$$\frac{s_r f(Ar)}{r} \simeq \ln r.$$

This example fails to be globally local with respect to $f(r) = Cr^2 \ln r$ (roughly by a factor of $\ln r$), and a Tichonov type solution is present.

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Thank you very much!