A complex function theory useful in Mellin analysis. Applications

Carlo Bardaro

Department of Mathematics and Computer Science - University of Perugia

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A joint research with

Paul L. Butzer (RWTH-Aachen, Germany)

Ilaria Mantellini (DMI, University of Perugia, Italy)

Gerhard Schmeisser (Department Mathematik, FAU Erlangen-Nürnberg, Germany)

Bernstein spaces in L^p, $p \in [1, +\infty]$: B^p_σ denotes the space of all continuous L^p-functions $f : \mathbb{R} \to \mathbb{C}$ which have an extension to the whole complex plane as an entire function of exponential type σ, i.e.

$$|f(z)| \leq C \exp(\sigma |\Im z|) \quad (z \in \mathbb{C}).$$

Paley-Wiener theorem in L^p, $p \in [1, 2]$: $f \in B^p_\sigma$ if and only if $\widehat{f}(v) = 0$ for a.e. $|v| > \sigma$.

- R.P. Boas, Entire functions, (1954).

The Sampling Theorem: If $f \in B^p_\sigma$, $p \in [1,2]$, then

$$f(z) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma z}{\pi} - k\right) \quad (z \in \mathbb{C}),$$

where

$$\operatorname{sinc}(z) := \frac{\sin(\pi z)}{\pi z}, \text{ for } z \neq 0, \text{and } \operatorname{sinc}(0) = 1.$$

- J.R. Higgins, Sampling Theory in Fourier and Signal Analysis, (1996).

Main references in applications:

- N.Ostrowski - D.Sornette - P.Parker - E.R.Pike (1980),
- M. Bertero - E.R. Pike (1991),
- F. Gori (1993).

Original inverse problem:

$$g(t) = \int_0^\infty K(t,s) f(s) ds,$$

g is the data function, K is a kernel, f is the unknown function.
Example: Polydispersity analysis by photon correlation spectroscopy, $K(x) := \exp(-x)$. Solution $(t_n = \exp(\pi n/\Omega), \quad n = 0, \pm 1, \pm 2, \ldots)$

$$f(t) = \sum_{k \in \mathbb{Z}} f(t_n) S(\Omega, t/t_n), \quad S(\Omega, t) = \frac{1}{\sqrt{t}} \frac{\sin(\Omega \log t)}{\Omega \log t}.$$

Main tool: Mellin transform theory and Mellin analysis.

A rigorous treatment via Mellin transforms:

- R.G. Mamedov - G.N. Orudhzev, Baku, 1979-1981
- R.G. Mamedov, Baku, 1991
- P.L. Butzer-S. Jansche (1997-2000)

For a given $c \in \mathbb{R}$ we put

$$X_c := \{f : \mathbb{R}^+ \to \mathbb{C} : (\cdot)^{c-1} f(\cdot) \in L^1(\mathbb{R}^+)\} \quad \|f\|_{X_c} := \|(\cdot)^{c-1} f(\cdot)\|_1$$

The Mellin Transform: for $f \in X_c$ we define

$$M[f](s) = [f]^\wedge_M(s) = \int_0^{+\infty} f(t) t^{s-1} dt, \quad s = c + it, \ t \in \mathbb{R}$$

$$|[f]^\wedge_M(c+it)| \leq \|f\|_{X_c}.$$

Mellin bandlimited functions: for $T > 0$, we define

$$\widehat{B}_{c,T} := \{f \in X_c \cap C(\mathbb{R}^+) : [f]^\wedge_M(c+it) = 0, \text{ for all } |t| > T\}.$$

For $1 < p < \infty$, $X_c^p := \{f : \mathbb{R}^+ \to \mathbb{C} : (\cdot)^{c-1/p} f(\cdot) \in L^p(\mathbb{R}^+)\}$

$$\|f\|_{X_c^p} := \|(\cdot)^{c-1/p} f(\cdot)\|_p.$$

For $p = \infty$ we set $\|f\|_{X_c^\infty} := \sup_{x \in \mathbb{R}^+} |x^c f(x)|$.

The Mellin Transform in X_c^2: for $f \in X_c^2$, $s = c + it$,

$$\lim_{\rho \to +\infty} \left\| M^2[f](s) - \int_{1/\rho}^{\rho} f(u) u^{s-1} du \right\|_{L^2(c+i\mathbb{R})} = 0.$$

Mellin bandlimited functions in X_c^2: for $T > 0$, we define

$$\widehat{B}^2_{c,T} := \{f \in X_c^2 \cap C(\mathbb{R}^+) : [f]^{\wedge}_{M^2}(c + it) = 0, \text{ a.e. } |t| > T\}, \ \widehat{B}_{c,T} \subset \widehat{B}^2_{c,T}.$$

- P.L. Butzer - S. Jansche, 1999

The Inverse Mellin Transform in X_c: for $g \in L^1(c + i\mathbb{R})$,

$$M_c^{-1}[g](x) \equiv M_c^{-1}[g(c+it)](x) := \frac{x^{-c}}{2\pi} \int_{-\infty}^{\infty} g(c+it) x^{-it} dt, \quad (x \in \mathbb{R}^+).$$

The Inverse Mellin Transform in X_c^2: for $g \in L^2(c + i\mathbb{R})$,

$$\lim_{\rho \to \infty} \left\| M_c^{2,-1}[g](x) - \frac{1}{2\pi} \int_{-\rho}^{\rho} g(c+it) x^{-c-it} dt \right\|_{X_c^2} = 0.$$

- P.L. Butzer - S. Jansche, 1999.

The pointwise Mellin differential operator: for $f : \mathbb{R}^+ \to \mathbb{C}$ and for $c \in \mathbb{R}$ we define

$$\Theta_c^1 f(x) \equiv \Theta_c f(x) := xf'(x) + cf(x), \qquad \Theta_c^r = \Theta_c(\Theta_c^{r-1}).$$

A representation theorem

$$(\Theta_c^r f)(x) = \sum_{k=0}^{r} S_c(r,k) x^k f^{(k)}(x)$$

- P.L. Butzer - S. Jansche, 1999.
- Other contributors to Mellin analysis: G. Vinti, L. Angeloni, L. Zampogni, D. Costarelli, A. Sambucini, A. Boccuto, M. Seracini

Exponential Sampling Theorem
(Butzer -Jansche, 1998, 1999)
Let $T > 0$ and $f \in \widehat{B}_{c,\pi T}$, then for any $x \in \mathbb{R}^+$

$$f(x) = \sum_{k=-\infty}^{\infty} f(e^{k/T}) \mathrm{lin}_{c/T}(e^{-k} x^T),$$

where

$$\mathrm{lin}_c(x) := \frac{x^{-c}}{2\pi i} \frac{x^{\pi i} - x^{-\pi i}}{\log x} = x^{-c} \mathrm{sinc}(\log x), \; x \in \mathbb{R}^+ \setminus \{1\},$$

and $\mathrm{lin}_c(1) := 1$
For the L^2-version of EST see

- P.L. Butzer - S. Jansche, 1999

The starting point

Theorem(BBMS, JAT 2016)

Let $f \in \widehat{B}^2_{c,T}$ be a non zero function. Then the function $g(x) := x^c f(x)$ cannot be extended to the whole complex plane as an entire function.

Mellin-Bernstein spaces

$\widetilde{B}^2_{c,T}$ contains all functions $f \in X^2_c$ s.t. g has an analytic extension on the Riemann surface $S_{\log}$ with branches g_k s.t.

- $|g_k(re^{i\theta})| \leq Ce^{T|2\pi k+\theta|}$
- $\lim_{r\to 0} g_k(re^{i\theta}) = \lim_{r\to+\infty} g_k(re^{i\theta}) = 0$ unif. w.r. θ.

Paley-Wiener Theorem (BBMS, JAT 2016)

$$\widetilde{B}^2_{c,T} = \widehat{B}^2_{c,T}.$$

Let us denote $\mathbb{H} := \{(r,\theta) \in \mathbb{R}^+ \times \mathbb{R}\}$.
Definition If $\mathcal{D} \subset \mathbb{H}$ we say that $f : \mathcal{D} \to \mathbb{C}$ is polar-analytic on $\mathcal{D}$ if for any $(r_0, \theta_0) \in \mathcal{D}$ the limit exists

$$\lim_{(r,\theta)\to(r_0,\theta_0)} \frac{f(r,\theta) - f(r_0,\theta_0)}{re^{i\theta} - r_0 e^{i\theta_0}} =: (D_{\mathrm{pol}} f)(r_0, \theta_0).$$

Definition. The polar Mellin derivative of f is defined by
$\widetilde{\Theta}_c f(r,\theta) := re^{i\theta}(D_{\mathrm{pol}} f)(r,\theta) + cf(r,\theta)$.

$f = u + iv$, $u, v : \mathcal{D} \to \mathbb{C}$ verifies

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}, \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

Expressions of the polar derivative

$$(D_{\mathrm{pol}} f)(r,\theta) = e^{-i\theta}\left(\frac{\partial}{\partial r}u(r,\theta) + i\frac{\partial}{\partial r}v(r,\theta)\right)$$

$$(D_{\mathrm{pol}} f)(r,\theta) = \frac{e^{-i\theta}}{r}\left(\frac{\partial}{\partial \theta}v(r,\theta) - i\frac{\partial}{\partial \theta}u(r,\theta)\right)$$

$$(D_{\mathrm{pol}} f)(r,\theta) = e^{-i\theta}\,\frac{\partial f}{\partial r}(r,\theta) = \frac{e^{-i\theta}}{ir}\,\frac{\partial f}{\partial \theta}(r,\theta).$$

Moreover If $\varphi(\cdot) := f(\cdot, 0)$ then $(D_{\mathrm{pol}} f)(r,0) = \varphi'(r)$ and $\widetilde{\Theta}_c f(r,0) = \Theta_c \varphi(r)$.

Definition. For $c \in \mathbb{R}$, $T > 0$ and $p \in [1, +\infty]$ the *Mellin–Bernstein space* $\mathscr{B}^p_{c,T}$ comprises all functions $f : \mathbb{H} \to \mathbb{C}$ with the following properties:

(i) f is polar-analytic on $\mathbb{H}$;

(ii) $f(\cdot, 0) \in X^p_c$;

(iii) there exists a positive constant C_f such that

$$|f(r,\theta)| \le C_f r^{-c} e^{T|\theta|} \qquad ((r,\theta) \in \mathbb{H}).$$

Theorem (BBMS, MATH. NACHR. 2017). Let $f \in \mathscr{B}^p_{c,T}$. Then

- $f(\cdot,\theta) \in X^p_c$, and $\|f(\cdot,\theta)\|_{X^p_c} \le e^{T|\theta|} \|f(\cdot,0)\|_{X^p_c}$
- $\lim_{r\to 0} = \lim_{r\to+\infty} r^c f(r,\theta) = 0$ unif. w.r. θ

Theorem (Paley-Wiener, BBMS, MATH.NACHR., 2017)
$\varphi \in X_c^2$ belongs to the Paley–Wiener space $\widehat{B}^2_{c,T}$ if and only if there exists a function $f \in \mathscr{B}^2_{c,T}$ such that $f(\cdot, 0) = \varphi(\cdot)$.

Definition. Let $(r_0, \theta_0) \in \mathbb{H}$ and $\rho > 0$.
The polar disk centered at (r_0, θ_0) with radius ρ is defined as:

$$E((r_0,\theta_0),\rho) := \left\{(r,\theta) \in \mathbb{H} : \big(\log(r/r_0)\big)^2 + (\theta-\theta_0)^2 < \rho^2\right\}$$

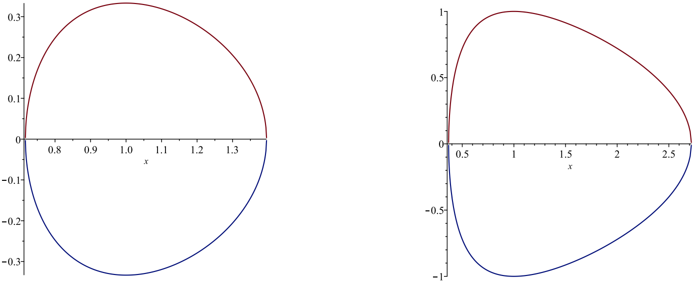


Figure: The polar-disks around the point $(r_0, \theta_0) = (1, 0)$ with $\rho = 1/3$ and $\rho = 1$.

Theorem. Let $\mathcal{D} \subset \mathbb{H}$ be a domain and let $f : \mathcal{D} \to \mathbb{C}$ be polar analytic on $\mathcal{D}$. Let $(r_0, \theta_0) \in \mathcal{D}$ and $c \in \mathbb{R}$. Then there holds the expansion

$$(re^{i\theta})^c f(r,\theta) = (r_0 e^{i\theta_0})^c \sum_{k=0}^{\infty} \frac{(\widetilde{\Theta}_c^k f)(r_0, \theta_0)}{k!} \big(\log(r/r_0) + i(\theta - \theta_0)\big)^k,$$

converging uniformly on every closed polar disk $E((r_0, \theta_0), \rho) \subset \mathcal{D}$.

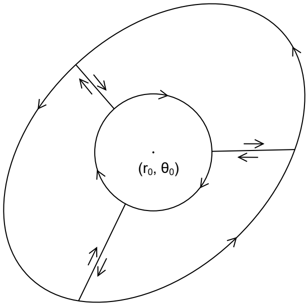
Corollary. If $(r_0, \theta_0) \in \mathcal{D}$ is an accumulation point of distinct zeros of f, then f is identically zero on $\mathcal{D}$.

Theorem. Let $\mathcal{D}$ be a convex domain in $\mathbb{H}$ and let $f : \mathcal{D} \to \mathbb{C}$ be polar-analytic on $\mathcal{D}$. Let γ be a positively oriented, closed, regular curve that is the boundary of a simply connected domain $\mathrm{int}\,(\gamma) \subset \mathcal{D}$. Then for $(r_0, \theta_0) \in \mathrm{int}\,(\gamma)$, $c \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{(re^{i\theta})^{c-1} f(r,\theta) e^{i\theta}}{\big(\log(r/r_0) + i(\theta - \theta_0)\big)^{k+1}}(dr + ird\theta) = (r_0 e^{i\theta_0})^c \frac{(\widetilde{\Theta}_c^k f)(r_0, \theta_0)}{k!}.$$

Proof. Let us consider the function

$$F(r,\theta) := \frac{(re^{i\theta})^{c-1} f(r,\theta)}{\left(\log(r/r_0) + i(\theta - \theta_0)\right)^{k+1}} \qquad ((r,\theta) \neq (r_0,\theta_0)).$$



Definition. Let $(r_0, \theta_0) \in \mathbb{H}$ and let $U \subset \mathbb{H}$ be open nhood of (r_0, θ_0). If $f : U \setminus \{(r_0, \theta_0)\} \to \mathbb{C}$ is polar analytic, then (r_0, θ_0) will be called an *isolated singularity*. An isolated singularity (r_0, θ_0) is called a *pole of order k*, if there exists a polar analytic function $g : U \to \mathbb{C}$ with $g(r_0, \theta_0) \neq 0$ such that

$$f(r, \theta) = \frac{g(r, \theta)}{\big(\log(r/r_0) + i(\theta - \theta_0)\big)^k} \qquad ((r, \theta) \neq (r_0, \theta_0)).$$

In this case the *c-residue* of f at (r_0, θ_0) is defined as

$$(\mathrm{res}_c f)(r_0, \theta_0) := (r_0 e^{i\theta_0})^c \frac{(\widetilde{\Theta}_c^{k-1} g)(r_0, \theta_0)}{(k-1)!}.$$

Theorem. Let $\mathcal{D} \subset \mathbb{H}$ be a domain in $\mathbb{H}$ and let f be polar-anaytic on $\mathcal{D}$ except for isolated singularities which are all poles. Let γ be a positively oriented, closed, regular curve that is the boundary of a simply connected domain $\mathrm{int}\,(\gamma) \subset \mathcal{D}$. Suppose that no isolated singularity lies on γ, while (r_j, θ_j) for $j = 1, \ldots, m$ are the singularities lying in $\mathrm{int}\,(\gamma)$. Then, for $c \in \mathbb{R}$, there holds

$$\int_\gamma (re^{i\theta})^{c-1} f(r,\theta) e^{i\theta}(dr + ird\theta) = 2\pi i \sum_{j=1}^{m} (\mathrm{res}_c f)(r_j, \theta_j).$$

Theorem. Let $T > 0$, $f \in \mathscr{B}^2_{c,T}$, $c \in \mathbb{R}$. Then

$$f(r,\theta) = \sum_{k\in\mathbb{Z}} f(e^{k\pi/T},\theta)\,\mathrm{lin}_{c\pi/T}(e^{-k}r^T).$$

Multiplied by r^c the series converges absolutely and uniformly on strips of bounded width parallel to the r-axis.

In particular, setting $\varphi(r) := f(r,0)$ one has

$$\varphi(r) = \sum_{k\in\mathbb{Z}} \varphi(e^{k\pi/T})\,\mathrm{lin}_{c\pi/T}(e^{-k}r^T).$$

Theorem Let $f \in \mathscr{B}_c^p$, where $p \in [1, \infty]$, $c \in \mathbb{R}$, and $T > 0$. Then

$$(\widetilde{\Theta}_c f)(r, \theta) = \frac{4T}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(2k+1)^2} e^{(k+1/2)\pi c/T} f(re^{(k+1/2)\pi/T}, \theta)$$

for $(r, \theta) \in \mathbb{H}$. Multiplied by r^c the series converges absolutely and uniformly on strips of bounded width parallel to the r-axis.

In particular, setting $\varphi(r) := f(r, 0)$ one has

$$(\Theta_c \varphi)(r) = \frac{4T}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(2k+1)^2} e^{(k+1/2)\pi c/T} \varphi(re^{(k+1/2)\pi/T}).$$

For Fourier case:

- P.L. Butzer - G. Schmeisser - R.L. Stens, (2013).

Corollary. Let $f \in \mathscr{B}_c^p$, where $p \in [1, \infty]$, $c \in \mathbb{R}$, and $T > 0$. Then for any $\theta \in \mathbb{R}$, we have

$$\|(\widetilde{\Theta}_c f)(\cdot, \theta)\|_{X_c^p} \leq T \|f(\cdot, \theta)\|_{X_c^p}.$$

Theorem. Let $f \in \mathscr{B}_c^\infty$, where $c \in \mathbb{R}$, and $T > 0$. Then for any $r \in \mathbb{R}^+$, we have

$$\begin{aligned} r^c f(r,0) &= \sin(T \log r) \Bigg[\frac{(\widetilde{\Theta}_c f)(1,0)}{T} + \frac{f(1,0)}{T \log r} \\ &+ T \log r \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^{k+1} e^{k\pi c/T} f(e^{k\pi/T}, 0)}{k\pi(k\pi - T \log r)} \Bigg]. \end{aligned}$$

The series converges absolutely and uniformly on compact subsets of $\mathbb{R}^+$.

For Fourier case:

- P.L. Butzer - G. Schmeisser - R.L. Stens, (2013).

For $a > 0$ we denote $\mathbb{H}_a := \{(r, \theta) : r > 0,\ \theta \in]-a, a[\}$.
Definition. Let $c \in \mathbb{R}, p \in [1, +\infty[, a > 0$.
The Mellin-Hardy space $H^p_c(\mathbb{H}_a)$ comprises all functions $f : \mathbb{H}_a \to \mathbb{C}$ that satisfy the following conditions:

(i) f is polar-analytic on $\mathbb{H}_a$;

(ii) $f(\cdot, \theta) \in X^p_c$ for each $\theta \in]-a, a[$;

(iii) there holds

$$\|f\|_{H^p_c(\mathbb{H}_a)} := \sup_{0<\theta<a} \left(\frac{\|f(\cdot, \theta)\|^p_{X^p_c} + \|f(\cdot, -\theta)\|^p_{X^p_c}}{2} \right)^{1/p} < +\infty.$$

When $a \in]0, \pi]$ we can associate with each function $f \in H^p_c(\mathbb{H}_a)$ a function g analytic on the sector $\mathcal{S}_a := \{z \in \mathbb{C} : |\arg z| < a\}$ by defining $g(re^{i\theta}) := f(r, \theta)$. The collection of all such functions constitutes a Hardy-type space $H^p_c(\mathcal{S}_a)$, which may be identified with $H^p_c(\mathbb{H}_a)$.

Mellin-Poisson summation formula (Butzer-Jansche, 1998)
If $f \in W_c^{1,1}(\mathbb{R}^+)$ be a continuous function on $\mathbb{R}^+$ such that

$$\sum_{k\in\mathbb{Z}} |[f]^{\wedge}_{M_c}(c + 2\pi i\sigma k)| < \infty.$$

Then

$$\sum_{k\in\mathbb{Z}} f(e^{k/\sigma})e^{kc/\sigma} = \sigma \sum_{k\in\mathbb{Z}} [f]^{\wedge}_{M_c}(c + 2\pi i\sigma k).$$

If $f \in \widehat{B}^1_{c,2\pi\sigma}$, then the above equality reduces to

$$\int_0^{\infty} f(x)x^{c-1}dx = \frac{1}{\sigma}\sum_{k=-\infty}^{\infty} f(e^{k/\sigma})e^{kc/\sigma}.$$

If $f \notin \widehat{B}^1_{c,2\pi\sigma}$, we obtain the approximate quadrature formula

$$\int_0^{\infty} f(x)x^{c-1}dx = \frac{1}{\sigma}\sum_{k=-\infty}^{\infty} f(e^{k/\sigma})e^{kc/\sigma} + R_{c,\sigma}[f].$$

Mellin inversion class. Let us denote by $\mathcal{M}_c^p$ the class of all functions $f \in X_c^p \cap C(\mathbb{R}^+)$ such that $[f]^\wedge_{M_c^p} \in L^1(c + i\mathbb{R})$.

Mellin-even and Mellin-odd parts (Schmeisser, 1999)

For $x > 0$ define:

$$f_{c+}(x) := \frac{1}{2}(x^c f(x) + x^{-c} f(1/x)), \quad f_{c-}(x) := \frac{1}{2}(x^c f(x) - x^{-c} f(1/x)),$$

so that $f(x) = x^{-c}(f_{c+}(x) + f_{c-}(x))$ for every $x > 0$.

Theorem (BBMS, Calcolo, 2018). Let $f \in \mathcal{M}_c^1$. Then for $a > 0$, we have for $\sigma \to +\infty$,

$$f \in H_0^*(\mathbb{H}_a), f(\cdot, 0) \equiv \varphi_{c+} \Longleftrightarrow R_{c,\sigma}[\varphi] = \mathcal{O}(e^{-2\pi a\sigma}).$$

Mellin translation operator (Butzer-Jansche, JFAA, 1997)
For $f : \mathbb{R}^+ \to \mathbb{C}$ and $h > 0$, define: $(\tau_h^c f)(x) := h^c f(hx)$.

Theorem (BBMS, Calcolo, 2018). Let $f \in \mathcal{M}_c^1$. Then for $a > 0$, we have for $\sigma \to +\infty$,

$$f \in H_0^*(\mathbb{H}_a), f(\cdot, 0) \equiv \varphi \Longleftrightarrow R_{c,\sigma}[\tau_h^c \varphi] = \mathcal{O}(e^{-2\pi a \sigma})$$

uniformly w.r.$h \in [e^{-1/(2\sigma)}, e^{1/(2\sigma)}]$.

- F. Stenger, 1973
- D.P. Dryanov, Q.I. Rahman - G. Schmeisser, 1990.
- W. Gautschi, 1991
- G. Schmeisser, 1999
- G. Mastroianni–G. Monegato, 2003
- G. Mastroianni–I. Notarangelo–G.V. Milovanovic, 2014.

THANKS FOR YOUR ATTENTION!