Mixed interpolatory and inference for non-intrusive reduced order nonlinear modelling

... so-called **MII** or ... $\{\textbf{SVD} - \textbf{SVD} - \textbf{SVD}\}$ algorithm

8th European Congress of Mathematics (Portorož, Slovenia)

"Rational approximation for data-driven modelling and complexity reduction of linear and nonlinear dynamical systems",
invited by S. Lefteriu and I.V. Gosea

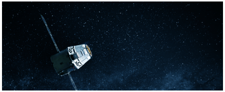
Charles Poussot-Vassal
June, 2021

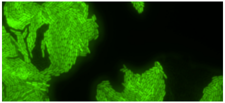
ONERA

THE FRENCH AEROSPACE LAB

Dynamical models are centrals tools in engineering

- **for verification and validation**
  (μ, $\mathcal{H}_\infty$-norm, pseudo-spectra, Monte Carlo)
  » `M. Voigt & W. Schilders [8ECM]`
- **uncertainty propagation**
  (Multi Disc. Optim., robust optim.)
- **control synthesis**
  ($\mathcal{H}_\infty/\mathcal{H}_2$-norm, MPC, adaptive)
  » `P. Kergus [8ECM]`





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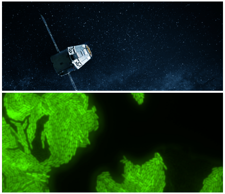

Complex models

- important sim. time
- memory burden
- inaccurate results
- limit model class

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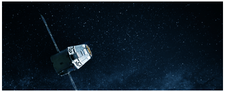

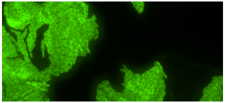

Complex models

- important sim. time
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$\Rightarrow$

Model simplification

Simplified models

- reduced sim. time
- memory saving
- accurate results
- rational model

Forewords

Motivating problem: Météo France pollutants plume dispersion

Simulation for ecological and civilian safety issues

- **How to predict the pollutants plume dispersion episodes**
- **How to organise emergency plans?**
- **How to organise buildings in cities?**
- Météo France & CERFACS & ONERA use-case
- 5,700 hours of simulation for 3 hours of prediction

Intergovernmental Panel on Climate Change (IPCC), *"https://www.ipcc.ch/report/sixth-assessment-report-cycle/"*, 6th report (to be published in 2022).

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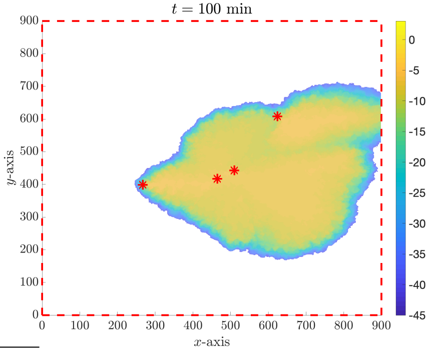
LES... a dynamical system
Over a fictive airport map



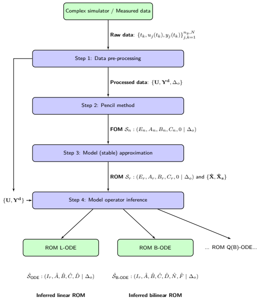
📚 Intergovernmental Panel on Climate Change (IPCC), *"https://www.ipcc.ch/report/sixth-assessment-report-cycle/"*, 6th report (to be published in 2022).

Fluids, pollutants and NL-ROM

- P. Vuillemin [approx.]
- T. Sabatier [pollutants dyn.]
- C. Sarrat [large eddy simu.]
- Colleagues of CERFACS [CPU facility]

Development

Mixed Interpolatory and Inference
is an hybrid approach developed for
nonlinear ROM construction



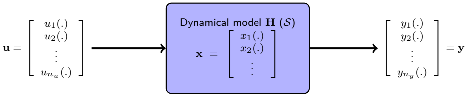
C. P-V., T. Sabatier, C. Sarrat, P. Vuillemin, *"Mixed interpolatory and inference non-intrusive reduced order modeling with application to pollutants dispersion"*, https://arxiv.org/abs/2012.07126.

Dynamical systems and interpolation

Model & Data $\Rightarrow$ Rational functions

(Continuous time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Continuous frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



(Sampled time-domain) $\{t_i, \mathbf{G}(t_i)\}_{i=1}^N$

(Sampled frequency-domain) $\{z_i, \mathbf{G}(z_i)\}_{i=1}^N$

Dynamical systems and interpolation

Model & Data $\Rightarrow$ Rational functions

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE

- L-DDE
- L-PDE

- B-DAE
- Q-DAE

- Data (time)
- Data (frequency)
- Data (parametric)

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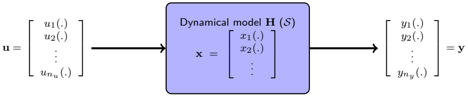

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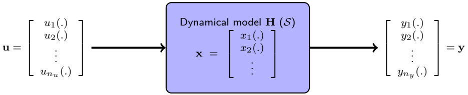

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(Continuous frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$

Properties

- Rational transfer function $\mathbf{H}$
- Finite realisation $\mathbf{dim}(\mathcal{S}) = n$
- Finite number of singularities $\Lambda(\mathcal{S}) = n$

$$\mathbf{H}(s) = \frac{2}{s+1} \text{ or } \mathbf{H}(s) = \frac{2s+4}{s+1} \text{ or } \mathbf{H}(s) = \frac{s^2+s+1}{s+1}$$

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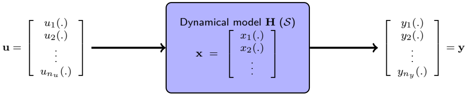

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(Continuous frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$

Properties

- Non-rational transfer function $\mathbf{H}$
- Finite/infinite realisation $\mathbf{dim}(\mathcal{S}) = n, \infty$
- Infinite number of singularities $\Lambda(\mathcal{S}) = \infty$

$$\mathbf{H}(s) = \frac{1}{s + e^{-s}} e^{-2s}$$

» P. Schulze [8ECM]

Dynamical systems and interpolation

Model & Data $\Rightarrow$ Rational functions

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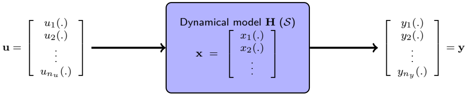

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(Continuous frequency-domain) $\mathbf{H} \sim \mathbf{u} \to \mathbf{y}$

Properties

- Multi-valued coupled transfer function $\mathbf{H}$
- Finite/infinite realisation $\mathbf{dim}(\mathcal{S}) = n, \infty$
- Singularities ?

$$
\begin{aligned}
\mathbf{H}_1(s_1) &= C\mathbf{\Phi}(s_1)B \\
\mathbf{H}_2(s_1,s_2) &= C\mathbf{\Phi}(s_2)N\mathbf{\Phi}(s_1)B \\
\mathbf{H}_3(s_1,s_2,s_3) &= C\mathbf{\Phi}(s_3)N\mathbf{\Phi}(s_2)N\mathbf{\Phi}(s_1)B \\
&\vdots
\end{aligned}
$$

» D. Karachalios [8ECM]

Dynamical systems and interpolation

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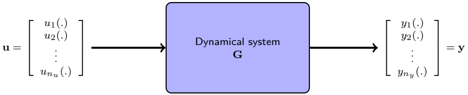
- **Data (frequency)**
- Data (parametric)



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(Sampled frequency-domain) $\{z_i, \mathbf{G}(z_i)\}_{i=1}^{N}$

Properties

- Hidden transfer functions / realisations
- Time / frequency sequence sets

$$\mathbf{G}(s) = \frac{5}{2s+1} = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)}$$

Known at complex data $\{\lambda_i, \mu_j\} = \{[1,3],[2,4]\}$

$$\{\mathbf{G}(\lambda_i), \mathbf{G}(\mu_j)\} = \{\mathbf{w}_i, \mathbf{v}_j\} = \{[5/3, 5/7], [1, 5/9]\}$$

» A.C. Antoulas [8ECM]

Rational functions. . . a key ingredient in engineering

Barycentric form (stable and **central in Antoulas, Anderson & Mayo landmark**)

$$\mathbf{H}(z) = \frac{\sum_i \beta_i/(z-\lambda_i)}{\sum_i \alpha_i/(z-\lambda_i)}$$

L.N. Trefethen, *"Rational functions (von Neumann Prize lecture)"*, SIAM Annual Meeting, 2020.

A.C. Antoulas, C. Beattie and S. Gugercin, *"Interpolatory methods for model reduction"*, SIAM Computational Science and Engineering, Philadelphia, 2020.

Dynamical systems and interpolation

Rational functions, models and approximation

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Support points

Any rational function can be written in the Barycentric form, for any support points λ_i.

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$\Longrightarrow$

Rational function simplification

Interpolation

This is the basis of rational interpolation, model approximation and model reduction tools.

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Dynamical systems and interpolation

Rational interpolation - linear finite order model

Large model

- Rational function
- $n = 48$ singularities

$\Rightarrow$

Rational approximation

ROM

- Rational function
- r state variables

LAH model $h = 10^{-2}$

$$\begin{array}{rcl} \mathbf{x}_{k+1} & = & A\mathbf{x}_k + B\mathbf{u}_k \\ \mathbf{y}_k & = & C\mathbf{x}_k \end{array}$$

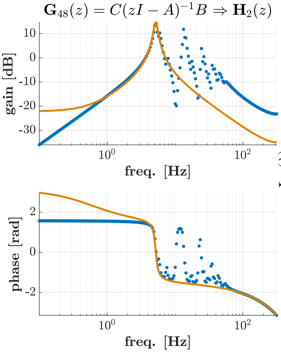
Support points

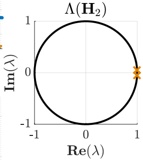
$$\lambda_i = e^{\imath \omega_i}$$

leads to

$$\frac{\sum_i \beta_i/(z-\lambda_i)}{\sum_i \alpha_i/(z-\lambda_i)}$$

(with post stabilisation)





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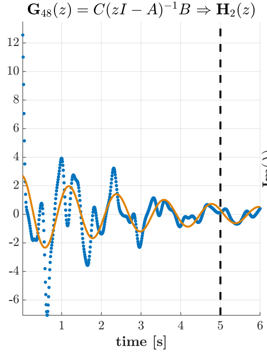

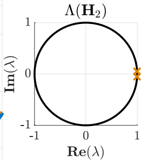

Dynamical systems and interpolation

Large delay model

- (Irrational) function
- Periodic singularities

$\Rightarrow$

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ROM

- Rational function
- r state variables

LAH model $h = 10^{-2}$

$$\begin{aligned} \mathbf{x}_{k+1} &= A\mathbf{x}_k - \mathbf{x}_{k-7} \\ &\quad + B\mathbf{u}_{k-30} \\ \mathbf{y}_k &= C\mathbf{x}_k \end{aligned}$$

Support points

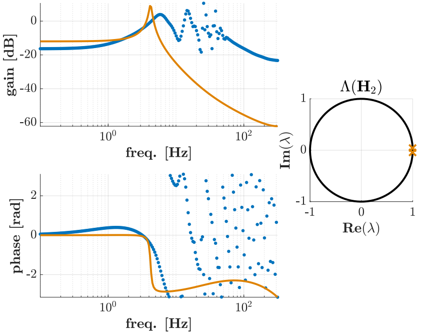
$$\lambda_i = e^{\imath\omega_i}$$

leads to

$$\frac{\sum_i \beta_i/(z-\lambda_i)}{\sum_i \alpha_i/(z-\lambda_i)}$$

(with post stabilisation)

$\mathbf{G}_{48}(z) = C(zI - (A - z^{-7}I))^{-1}Bz^{-30} \Rightarrow \mathbf{H}_2(z)$



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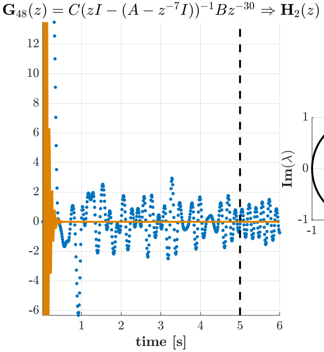

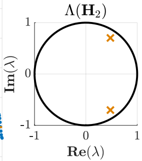

Dynamical systems and interpolation

Rational interpolation - nonlinear model

Large model
- Nonlinear function
- Singularities?

$\Longrightarrow$

MII

ROM
- Quad. bilin. function
- r state variables

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$$\begin{array}{rcl} \mathbf{x}_{k+1} & = & A\mathbf{x}_k + \mathbf{I}_n\mathbf{x}_k^2 + B\mathbf{u}_k \\ \mathbf{y}_k & = & C\mathbf{x}_k + \mathbf{1}_n\mathbf{x}_k^2 \end{array}$$

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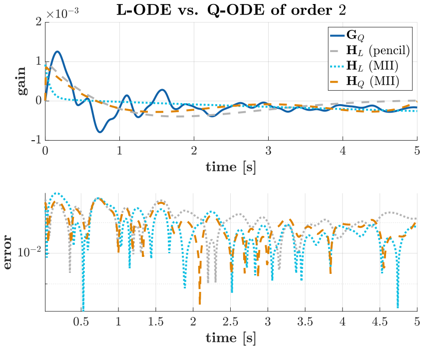
leading to

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \hat{A}\hat{\mathbf{x}}_k + \hat{B}\mathbf{u}_k \\ &\quad +\hat{Q}(\hat{\mathbf{x}}_k \otimes \hat{\mathbf{x}}_k) \\ \hat{\mathbf{y}}_k &= \hat{C}\hat{\mathbf{x}}_k \\ &\quad +\hat{G}(\hat{\mathbf{x}}_k \otimes \hat{\mathbf{x}}_k) \end{aligned}$$



Mixed Interpolatory and Inference

General idea

Merge interpolation and inference to learn a structured model

- Interpolatory pencil method [Antoulas/Ionita/Kamlan/...]
- Interpolatory Loewner method [Antoulas/Gugercin/Gosea/Lefteriu/Mayo/...]
- Operator inference [Benner/Brunton/Gosea/Peherstorfer/Willcox/...]

Pencil

- (Real)-domain time output data
- Ho-Kalman matrix
- **SVD**

Loewner

- (Complex)-domain freq. output data
- Loewner matrices
- **SVD**

Operator Inference

- (Real)-domain time input/state data
- Model structure
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A.C. Ionita and A.C. Antoulas, *"Matrix pencil in time and frequency domain identification"*, Springer chapter, 2016.

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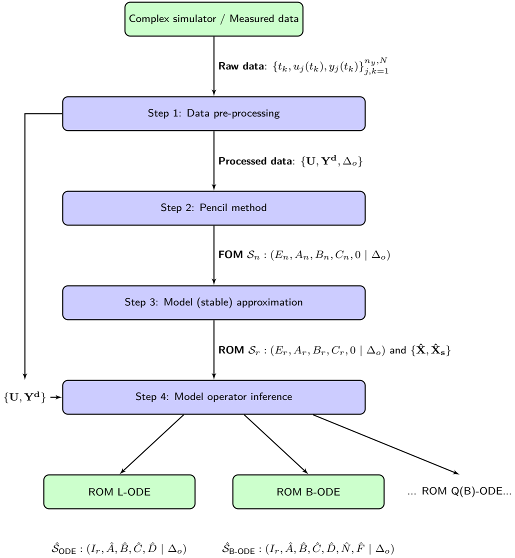
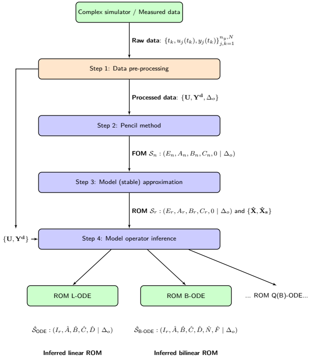
Mixed Interpolatory and Inference

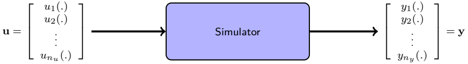
MII big picture



Mixed Interpolatory and Inference

Step 1: Pre-process data





Raw data

$$\mathbf{U} := \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_N \\ | & | & & | \end{bmatrix}$$

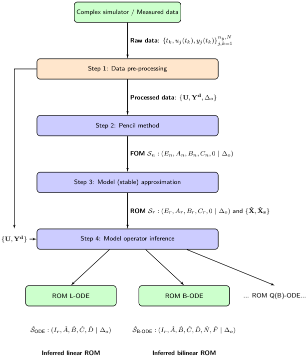
$$\mathbf{Y} := \begin{bmatrix} | & | & & | \\ \mathbf{y}_1 & \mathbf{y}_2 & \dots & \mathbf{y}_N \\ | & | & & | \end{bmatrix}$$

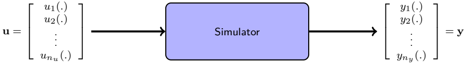
$\mathbf{X}$:= Not accessible

I. Pontes Duff, C. P-V. and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input/output-delay structured reduced order models"*, Systems & Control Letters, 2018.

Mixed Interpolatory and Inference

Step 1: Pre-process data





Raw data (delayed shifted)

$$\mathbf{U} := \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_N \\ | & | & & | \end{bmatrix}$$

$$\mathbf{Y^d} := \begin{bmatrix} | & | & & | \\ \mathbf{y}_1^d & \mathbf{y}_2^d & \dots & \mathbf{y}_N^d \\ | & | & & | \end{bmatrix}$$

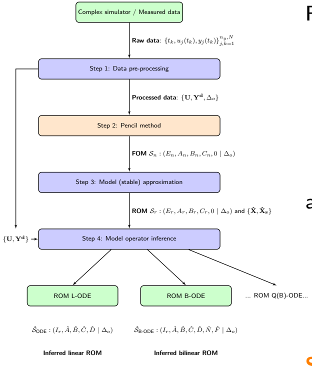
With delay operator $\Delta_o(\cdot)$ as

$$\begin{array}{rlcl} \Delta_o : & \mathbb{R}^{n_y} & \rightarrow & \mathbb{R}^{n_y} \\ & \mathbf{y}(t_k) & \rightarrow & \mathbf{y}(t_k - \tau) = \mathbf{y}^d(t_k), \end{array}$$

I. Pontes Duff, C. P-V. and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input/output-delay structured reduced order models"*, Systems & Control Letters, 2018.

Mixed Interpolatory and Inference

Step 2: Pencil



From $\mathbf{U}$ and $\mathbf{Y^d}$, construct for each i-th row

$$\mathcal{H}_i = \begin{bmatrix} \mathbf{y}(t_1) & \mathbf{y}(t_2) & \dots & \mathbf{y}(t_{n_i+1}) \\ \mathbf{y}(t_2) & \mathbf{y}(t_3) & \dots & \mathbf{y}(t_{n_i+2}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}(t_{n_i}) & \mathbf{y}(t_{n_i+1}) & \dots & \mathbf{y}(t_{2n_i}) \end{bmatrix}.$$

and select

$$\begin{aligned} E_{n_i} &= \mathcal{H}(1:n_i, 1:n_i) \\ A_{n_i} &= \mathcal{H}(1:n_i, 2:n_i+1) \\ B_{n_i} &= \mathcal{H}(1:n_i, 1) \\ C_{n_i} &= \mathcal{H}(1, 1:n_i) \end{aligned}$$

SVD leads to the i-th linear generating model

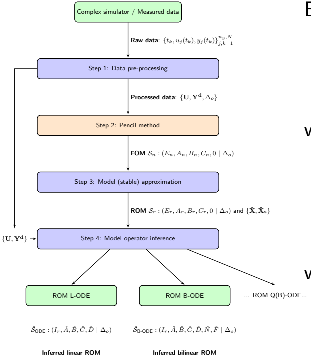
$$\mathcal{S}_{n_i} : (E_{n_i}, A_{n_i}, B_{n_i}, C_{n_i}, \mathbf{0}).$$

B. Ho and R. Kalman, *"Effective construction of linear state-variable models from input/output functions"*, Regelungs-technik, 1966.

A.C. Ionita and A.C. Antoulas, *"Matrix pencil in time and frequency domain identification"*, Springer chapter, 2016.

Mixed Interpolatory and Inference

Step 2: Pencil



By stacking $\mathcal{S}_{n_i}$ one gets

$$\begin{gathered}\mathcal{S} : (E, A, B, C, \mathbf{0}) \\ \mathcal{S}^d : (E, A, B, C, \mathbf{0} \mid \Delta_o)\end{gathered}$$

with

$$\begin{gathered}E = \mathbf{blkdiag}(E_{n_1}, \dots, E_{n_y}) \\ A = \mathbf{blkdiag}(A_{n_1}, \dots, A_{n_y}) \\ C = [C_{n_1}, \dots, C_{n_y}] \text{ and } B^T = [B_{n_1}^T, \dots, B_{n_y}^T]\end{gathered}$$

which associated transfer reads

$$\begin{gathered}\mathbf{H}(z) = C(zE - A)^{-1}B \\ \mathbf{H}^d(z) = \Delta_o(C)(zE - A)^{-1}B\end{gathered}$$

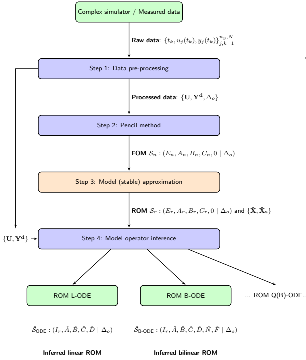
However, dimension rapidly grows with n_y...

B. Ho and R. Kalman, *"Effective construction of linear state-variable models from input/output functions"*, Regelungs-technik, 1966.

A.C. Ionita and A.C. Antoulas, *"Matrix pencil in time and frequency domain identification"*, Springer chapter, 2016.

Mixed Interpolatory and Inference

Step 3: Loewner



By $\mathbf{H}$ use Loewner as dynamic revealing tool, and interpolate over λ_i and μ_j with r-th order rational function obtained by **SVD**

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_k)}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_1)}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_k)}{\mu_q-\lambda_k} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{\mu_1\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)\lambda_1}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_k)\lambda_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_1)\lambda_1}{\mu_q-\lambda_1} & \cdots & \frac{\mu_q\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_k)\lambda_k}{\mu_q-\lambda_k} \end{bmatrix}$$

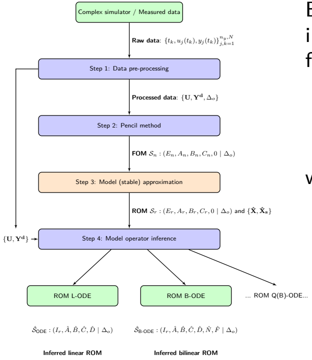
$$\mathbf{W} = \begin{bmatrix} \mathbf{H}(\lambda_1) & \ldots & \mathbf{H}(\lambda_k) \end{bmatrix}$$

$$\mathbf{V}^T = \begin{bmatrix} \mathbf{H}(\mu_1) & \ldots & \mathbf{H}(\mu_q) \end{bmatrix}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, 425(2-3), 2007, pp 634-662.

Mixed Interpolatory and Inference

Step 3: Loewner



By $\mathbf{H}$ use Loewner as dynamic revealing tool, and interpolate over λ_i and μ_j with r-th order rational function obtained by **SVD**

$$\begin{array}{rcl}\mathbf{H}_r(\mu_j) & = & \mathbf{H}(\mu_j) \\ \mathbf{H}_r(\lambda_i) & = & \mathbf{H}(\lambda_i)\end{array}$$

where

$$\mathbf{H}_r(z) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$$
$$\mathbf{H}_r^d(z) = \Delta_o(\mathbf{W})(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$$

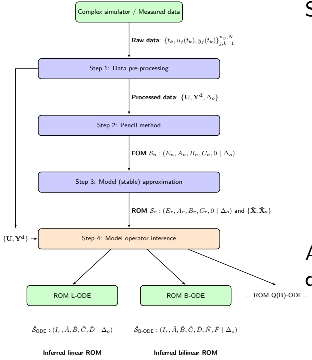
$$\mathcal{S}_r : (E_r, A_r, B_r, C_r)$$
$$\mathcal{S}_r^d : (E_r, A_r, B_r, C_r \mid \Delta_o)$$

However, $\mathbf{H}^d$ still linear...
Now we have access to the internal variables...

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, 425(2-3), 2007, pp 634-662.

Mixed Interpolatory and Inference

Step 4: Operator Inference



Simulate $\mathbf{H}_r$ with $\mathbf{U}$ and collect

$$\mathbf{X_r} := \begin{bmatrix} | & | & & | \\ \mathbf{x}_{r,1} & \mathbf{x}_{r,2} & \dots & \mathbf{x}_{r,N-1} \\ | & | & & | \end{bmatrix}$$

$$\mathbf{X_r^{(s)}} := \begin{bmatrix} | & | & & | \\ \mathbf{x}_{r,2} & \mathbf{x}_{r,3} & \dots & \mathbf{x}_{r,N} \\ | & | & & | \end{bmatrix}$$

And apply an **SVD (least square)** using original data and reduced simulated states.

B. Peherstorfer and K. Willcox, *"Data-driven operator inference for nonintrusive projection-based model reduction"*, Computer Methods in Applied Engineering, 2016.

I. Gosea and I. Pontes-Duff, *"Toward fitting structured nonlinear systems by means of dynamic mode decomposition"*, arXiv:2003.06484, 2020.

Mixed Interpolatory and Inference

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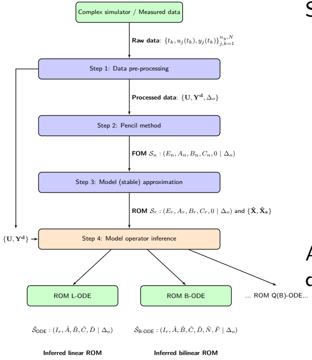

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And apply an **SVD (least square)** using original data and reduced simulated states.

$$\min_{\hat{A},\hat{B},\hat{C},\hat{D}} \left\| \begin{bmatrix} \mathbf{X_r^{(s)}} \\ \mathbf{Y^d} \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \mathbf{X_r} & \mathbf{U} \end{bmatrix} \right\|_F$$

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Mixed Interpolatory and Inference

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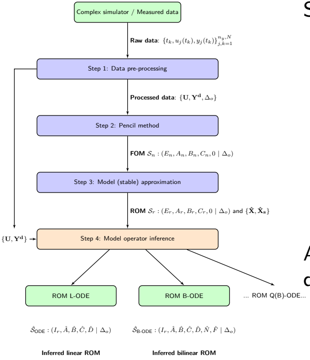

Simulate $\mathbf{H}_r$ with $\mathbf{U}$ and collect

$$\mathbf{X_r} := \begin{bmatrix} | & | & & | \\ \mathbf{x}_{r,1} & \mathbf{x}_{r,2} & \dots & \mathbf{x}_{r,N-1} \\ | & | & & | \end{bmatrix}$$

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$$\min_{\hat{A},\hat{B},\hat{C},\hat{D},\hat{N},\hat{F}} \left\| \begin{bmatrix} \mathbf{X_r^{(s)}} \\ \mathbf{Y^d} \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} & \hat{N} \\ \hat{C} & \hat{D} & \hat{F} \end{bmatrix} \begin{bmatrix} \mathbf{X_r} & \mathbf{U} & \mathbf{X_r}\mathbf{U} \end{bmatrix} \right\|_F$$

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Mixed Interpolatory and Inference

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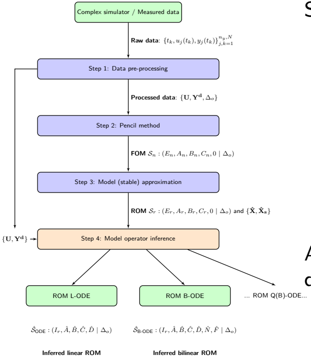

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$$\mathbf{X_r} := \begin{bmatrix} | & | & & | \\ \mathbf{x}_{r,1} & \mathbf{x}_{r,2} & \dots & \mathbf{x}_{r,N-1} \\ | & | & & | \end{bmatrix}$$

$$\mathbf{X_r^{(s)}} := \begin{bmatrix} | & | & & | \\ \mathbf{x}_{r,2} & \mathbf{x}_{r,3} & \dots & \mathbf{x}_{r,N} \\ | & | & & | \end{bmatrix}$$

And apply an **SVD (least square)** using original data and reduced simulated states.

$$\min_{\hat{A},\hat{B},\hat{C},\hat{D},\hat{Q},\hat{G}} \left\| \begin{bmatrix} \mathbf{X_r^{(s)}} \\ \mathbf{Y^d} \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} & \hat{Q} \\ \hat{C} & \hat{D} & \hat{G} \end{bmatrix} \begin{bmatrix} \mathbf{X_r} & \mathbf{U} & (\mathbf{X_r} \otimes \mathbf{X_r})H \end{bmatrix} \right\|_F$$

B. Peherstorfer and K. Willcox, *"Data-driven operator inference for nonintrusive projection-based model reduction"*, Computer Methods in Applied Engineering, 2016.

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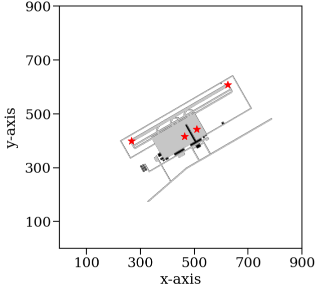
LES pollutant simulation

Météo France simulator

Numerical data

- **MESO-NH** simulator
  http://mesonh.aero.obs-mip.fr/mesonh54
- 3D Euler equations
- 900×900 grid points with a 10 meters horizontal resolution
- Wind from west to east
- 4 pollutant emission sources
- Dispersion of the plume
- Time data each 1-min
- Collection of $2,025$ grid points



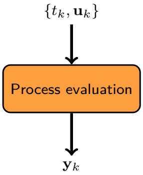
3h LES complex simulation
2,33 Go of data

LES pollutant simulation

Météo France simulator

Numerical data

- 3h LES complex simulation $\approx 7,500$h
- Time data over $2,025$ grid points



LES pollutant simulation

Météo France simulator

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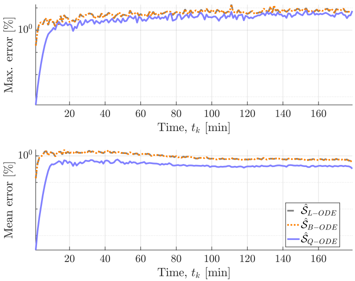
$\Longrightarrow$

MII

Accurate models

- $r = \{100, 30\}$
- **Q-ODE** or **B-ODE** model
- $\approx$ 1h (std. laptop)



Numerical data

- $r = \{\mathbf{100}, 30\}$
- No gain with **B-ODE**... expected due $\mathbf{U} = 1$
- Clear gain with **Q-ODE**... expected due to LES involving N&S

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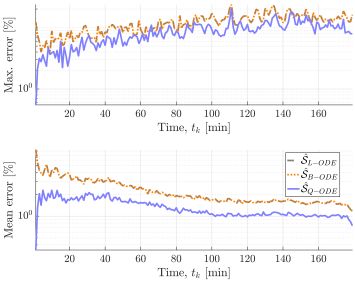

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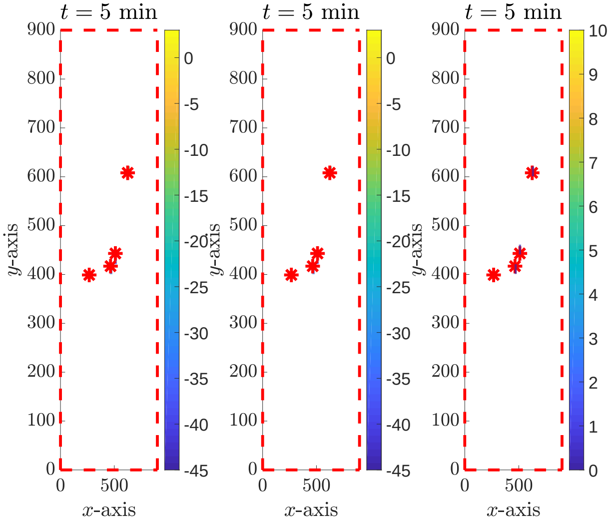
LES pollutant simulation

Météo France simulator LES vs. ROM vs. Relative error (%) (Grid $= 20$, $r = 100$)



Conclusions

MII... a versatile tool

MII merges interpolation and inference for reduced model construction,

- from time-domain data,
- with a structured (nonlinear) model,
- without knowledge of the state (saving memory),
- in a scalable and fast way.

$\rightarrow$ **direct impact in simulation engineers**
$\rightarrow$ **in practice MII is SVD-SVD-SVD**

C. P-V., T. Sabatier, C. Sarrat, P. Vuillemin, *"Mixed interpolatory and inference non-intrusive reduced order modeling with application to pollutants dispersion"*, https://arxiv.org/abs/2012.07126.

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Interesting results but

- tuning remains at some steps
- convergence is not well understood so far
- idea for improvement are welcome

C. P-V., T. Sabatier, C. Sarrat, P. Vuillemin, *"Mixed interpolatory and inference non-intrusive reduced order modeling with application to pollutants dispersion"*, https://arxiv.org/abs/2012.07126.

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- **Technical references and slides at**
  `sites.google.com/site/charlespoussotvassal/`
- **MOR Toolbox integrated tool at**
  `mordigitalsystems.fr/`

C. P-V., T. Sabatier, C. Sarrat, P. Vuillemin, *"Mixed interpolatory and inference non-intrusive reduced order modeling with application to pollutants dispersion"*, https://arxiv.org/abs/2012.07126.

Mixed interpolatory and inference for non-intrusive reduced order nonlinear modelling

... so-called **MII** or ... $\{\textbf{SVD} - \textbf{SVD} - \textbf{SVD}\}$ algorithm

8th European Congress of Mathematics (Portorož, Slovenia)

"Rational approximation for data-driven modelling and complexity reduction of linear and nonlinear dynamical systems",
invited by S. Lefteriu and I.V. Gosea

Charles Poussot-Vassal
June, 2021

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