On a construction of quaternionic and octonionic Riemann surfaces

(extract from [2]) by G. Gentili J. Prezelj and F. Vlacci

Introduction

Let $\mathbb{K}$ denote either the division algebra of quaternions $\mathbb{H}$ or that of octonions $\mathbb{O}$, and let $\mathbb{S} \subset \mathbb{K}$ be the 2-sphere or, respectively, the 6-sphere of imaginary units, i.e. the sets of $I \in \mathbb{K}$ such that $I^2 = -1$. If $I \in \mathbb{K}$ we define the *slice* $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ and say that a domain $\Omega \subset \mathbb{K}$ is a *slice domain* if $\Omega \cap \mathbb{R} \neq \varnothing$ and $\Omega_I := \Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$.

Let $\Omega \subseteq \mathbb{K}$ be a slice domain and let $f : \Omega \to \mathbb{K}$ be a function. If, for an imaginary unit I of $\mathbb{K}$, the restriction $f_I := f_{|\Omega_I}$ has continuous partial derivatives and

$$\bar{\partial}_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+yI) \equiv 0. \qquad (1)$$

then f_I is called *holomorphic*. If f_I is holomorphic for all imaginary units of $\mathbb{K}$, then the function f is called *slice regular*.

We refer the interested reader to the monograph [1] for an introduction to the main properties of slice regular functions in the quaternionic setting.

Main results

As customary, a differentiable map will be called *an immersion* if its differential is injective at all points of the domain of definition.

Let n, N be natural numbers with $N \geq n$ and let Ω be domain in $\mathbb{R}^n$. An at least C^1 immersion $f : \Omega \to \mathbb{R}^N$ will be called a *conformal or isothermal map* if the matrix of the differential of f is conformal, i.e., if it satisfies

$$^t df(p) df(p) = k(p) I_n \qquad (2)$$

for a (never vanishing and at least C^1) function $k : \Omega \to \mathbb{R}$

We specialize this definition for our purpouses.

Let Ω be a slice domain in $\mathbb{H} \cong \mathbb{R}^4$ and let $N \geq 4$ be a natural number. Let $f : \Omega \to \mathbb{R}^N$ be an at least C^1 immersion. If, for any $I \in \mathbb{S}$, $df_{|\mathbb{C}_I}$ and $df_{|\mathbb{C}_I^\perp}$ satisfy (2), then f will be called a *slice conformal or slice isothermal immersion.*

Notice that in general slice conformality does not imply conformality.

If f is an injective slice conformal immersion, then it will be called *slice conformal or slice isothermal parameterization* and $f(\Omega)$ in $\mathbb{R}^N$ will be called a *(parameterized) Riemann 4-manifold* of $\mathbb{R}^N$. In case $f : \Omega \to \mathbb{R}^N$ itself is a conformal map, then the parameterized 4-manifold $f(\Omega)$ in $\mathbb{R}^N$ will be called a *special (parameterized) Riemann 4-manifold* of $\mathbb{R}^N$.

Let $\Omega \subseteq \mathbb{H}$ be a slice domain; consider $F : \Omega \to \mathbb{H}^2 \cong \mathbb{R}^8$ where $F(q) = (f(q), g(q))$ with $f, g : \Omega \to \mathbb{H}$ slice regular functions. If F is an immersion, then F will be called *a slice regular curve (in $\mathbb{H}^2$)*. Furthermore, if F is injective, then $F(\Omega)$ is a parameterized Riemann 4-manifold in $\mathbb{H}^2$.

If $f : \Omega \to \mathbb{H}$ is any slice regular function, the slice regular curve $F : \Omega \to \mathbb{H}^2$ $F(q) := (q, f(q))$ is a slice conformal parameterization and the graph of f, i.e. $\Gamma(f) = \{(q, f(q)) : q \in \Omega\} \subseteq \mathbb{H} \times \mathbb{H}$ is a parameterized Riemann 4-manifold.

The Riemann 4-sphere

Let $f : \mathbb{R}^4 \cong \mathbb{H} \to \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5$ be

$$f(x+Iy) = \left(\frac{2(x+Iy)}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\right).$$

Then $df(x+Iy)_{|\mathbb{C}_I} =$

$$= \frac{2}{(1+x^2+y^2)^2}\begin{bmatrix} 1-x^2+y^2 & -2xy \\ -2xy & 1+x^2-y^2 \\ 0 & 0 \\ 0 & 0 \\ 2x & 2y \end{bmatrix}$$

$df(x+Iy)_{|\mathbb{C}_I^\perp} =$

$$= \frac{2}{(1+x^2+y^2)^2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1+x^2+y^2 & 0 \\ 0 & 1+x^2+y^2 \\ 0 & 0 \end{bmatrix}$$

The Helicoidal 4-manifold

Consider $g : \mathbb{R}^4 \cong \mathbb{H} \to \mathbb{H} \times \Im\mathbb{H} \cong \mathbb{R}^7$ with $g(x+Iy) =$

$$= (\sinh x \cos y + I \sinh x \sin y, Iy) =$$

$$= (sh(x)c(y) + Ish(x)s(y), Iy),$$

then

$$dg(x+Iy)_{|\mathbb{C}I} = \begin{bmatrix} ch(x)c(y) & -sh(x)s(y) \\ ch(x)s(y) & sh(x)c(y) \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$dg(x+Iy)_{|\mathbb{C}_I^\perp} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{sh(x)s(y)}{y} & 0 \\ 0 & \frac{sh(x)s(y)}{y} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $\mathbb{H}^+ = \{q \in \mathbb{H} : \Re\, q > 0\}$ put $\mathscr{E}^+ = g(\mathbb{H}^+)$, and $E(x+Iy) := (\exp(x+Iy), Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$ defines an immersion and a diffeomorphism between $\mathbb{H}$ and $\mathscr{E}^+$ such that $\pi \circ E = \exp$; then $L : \mathscr{E}^+ \subset \mathbb{H} \times \Im(\mathbb{H}) \to \mathbb{H}$ $L(q,p) = \log|q| + p$ is the $\mathscr{E}^+$-logarithm. Indeed, if $(q,p) \in \mathscr{E}^+$, then $q = |q| \exp p$ and so $E \circ L$ and $L \circ E$ are the identity.

References

[1] G. Gentili, C. Stoppato, D. C. Struppa, Regular Functions of a Quaternionic Variable, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013

[2] G. Gentili, J. Prezelj, F. Vlacci, *Slice conformality: Riemann manifolds and logarithm on quaternions and octonions* in `arXiv.org > math > math.CV`