Irregular solutions of the transport and Navier-Stokes equations

Maria Colombo

EPFL

EPFL SB, Institute of Mathematics

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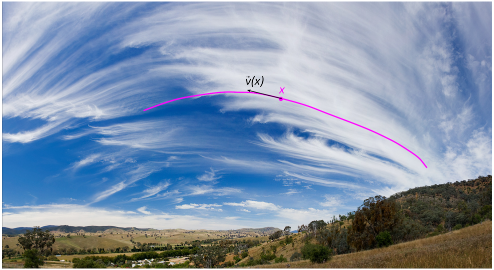
Incipit - Evolution of a single particle

We want to describe the motion of some particles of clouds. We model the clouds as a gas/fluid with given velocity $\mathbf{v}(x)$ for each position x (direction and intensity).
A single particle is transported along an integral curve of $\mathbf{v}$

$$\frac{d}{dt}\gamma(t) = \mathbf{v}(\gamma(t)) \qquad \text{for any } t \in [0, \infty).$$



Incipit - Evolution of a distribution of particles

If the particles are many, we model them as a distribution, namely with a measure μ_0. μ_t evolves according to the PDE

$$\partial_t \mu_t + \mathbf{v} \cdot \nabla \mu_t = 0.$$

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Outline of the talk

1. Flow of vector fields and continuity equation
2. Smooth vs nonsmooth theory
   - Lack of uniqueness of the flow for nonsmooth vector fields
   - Regular Lagrangian Flows and the nonsmooth theory
3. A.e. uniqueness of integral curves
   - Ambrosio's superposition principle
   - Ill-posedness of CE by convex integration

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The flow of a vector field

Given a vector field $\mathbf{b} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, consider the flow $\mathbf{X}$ of $\mathbf{b}$

$$\begin{cases} \frac{d}{dt}\mathbf{X}(t,x) = \mathbf{b}_t(\mathbf{X}(t,x)) & \forall t \in [0,\infty) \\ \mathbf{X}(0,x) = x. \end{cases}$$

It can be seen

- as a collection of trajectories $\mathbf{X}(\cdot, x)$ labelled by $x \in \mathbb{R}^d$;
- as a collection of diffeomorphisms $\mathbf{X}(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$.

Continuity/transport equation

Consider the related PDE, named continuity equation

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\mathbf{b}_t \mu_t) = 0 & \text{in } (0,\infty) \times \mathbb{R}^d \\ \mu_0 \text{ given.} \end{cases}$$

When $\mathbf{b}_t$ is sufficiently smooth and $\mu_t : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is a smooth function, all derivatives can be computed.
Much less is needed to give a distributional sense to the PDE (e.g. $\mathbf{b}_t$ bounded and μ_t finite measures).
When

$$\operatorname{div} \mathbf{b}_t \equiv 0,$$

the continuity equation is equivalent to the transport equation

$$\partial_t \mu_t + \mathbf{b} \cdot \nabla \mu_t = 0.$$

Connection between continuity equation and flows

Solutions of the CE flow along integral curves of b

Given **b**, its flow **X** an initial distribution of mass $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$, a solution of the CE is

$$\mu_t := \mathbf{X}(t,\cdot)_\# \mu_0.$$

Recall that the measure $\mathbf{X}(t,\cdot)_\#\mu_0$ is defined by

$$\int_{\mathbb{R}^d} \varphi(x)\, d[\mathbf{X}(t,\cdot)_\#\mu_0](x) = \int_{\mathbb{R}^d} \varphi(\mathbf{X}(t,x))\, d\mu_0(x) \quad \forall \varphi : \mathbb{R}^d \to \mathbb{R}.$$

Indeed, for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi\, d\mu_t = \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(\mathbf{X}(t,x))\, d\mu_0(x) = \int_{\mathbb{R}^d} \nabla\varphi(\mathbf{X}) \cdot \partial_t \mathbf{X}\, d\mu_0$$
$$= \int_{\mathbb{R}^d} \nabla\varphi(\mathbf{X}) \cdot \mathbf{b}_t(\mathbf{X})\, d\mu_0 = \int_{\mathbb{R}^d} \nabla\varphi \cdot \mathbf{b}_t\, d\mu_t.$$

This is the distributional formulation of the continuity equation.

Regularity of $\mathbf{b}$ matters

Is the solution of the continuity equation starting from μ_0 unique?

YES if $\nabla\mathbf{b}$ is bounded

Given a solution ν_t to CE, set $\widetilde{\nu}_t = \mathbf{X}(t,\cdot)^{-1}_{\#}\nu_t$ and compute

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi\, d\widetilde{\nu}_t = 0,$$

so $\mathbf{X}(t,\cdot)^{-1}_{\#}\nu_t = \widetilde{\nu}_t = \nu_0 = \mu_0 \quad \Rightarrow \quad \nu_t := \mathbf{X}(t,\cdot)_{\#}\mu_0.$

Cauchy-Lipschitz Theorem

Let $\mathbf{b}_t$ a vector field with $\nabla\mathbf{b}_t$ bounded. Then for every $x \in \mathbb{R}^d$ there exists a unique solution $\mathbf{X}(\cdot, x) : [0,\infty) \to \mathbb{R}^d$ of the ODE.

NO if $\mathbf{b}$ is less regular

As soon as uniqueness for the ODE fails.

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Why caring about less regular vector fields?

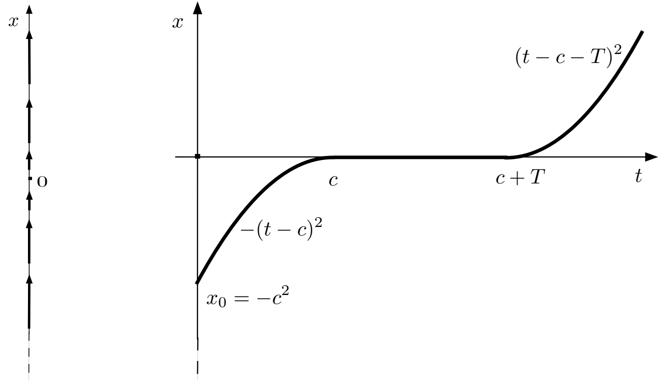
Less regular vector fields appear

- in fluid dynamics, when a fluid or a gas develop a turbulent behavior or a discontinuity/singularity (shear flows, shock waves...). As an example, in the theory of turbulence, the Onsager conjecture regards Holder continuous solutions to Euler; some of the optimal regularity estimates for Navier-Stokes are based on the understanding of its flow.
- in meteorology, to build solutions of the semigeostrophic system in 2d and 3d [Ambrosio, C., De Philippis, Figalli, '12, '14];
- in kinetic equations, to give a lagrangian description of solutions to the Vlasov-Poisson system [Ambrosio, C., Figalli, '15, '17];
- studying the geometry of nonsmooth manifolds with curvature bounds (in this direction, see also [C., Tione '20]).

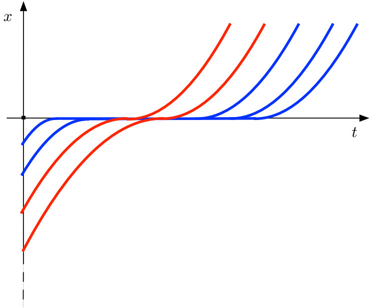
One-dimensional autonomous vector field with lack of uniqueness

$$\mathbf{b}(x) = 2\sqrt{|x|}, \qquad x \in \mathbb{R}$$

Given $x_0 = -c^2 < 0$, the 1-parameter family of curves that stop at the origin for an arbitrary time $T \geq 0$, solve the ODE.



Nonsmooth theory: lack of uniqueness



Between all the possible integral curves, a “better selection” could be made by the ones that do not stop in 0. In other words, we wish to select a collection of integral curves that “do not concentrate”.

Selection of a flow

Regular lagrangian flows

Given a vector field $\mathbf{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$, the map $\mathbf{X} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow of $\mathbf{b}$ if:

(i) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, $\mathbf{X}(\cdot, x)$ solves the ODE $\dot{x}(t) = \mathbf{b}_t(x(t))$ starting from x;

(ii) $\mathbf{X}(t, \cdot)_{\#}\mathscr{L}^d \le C\mathscr{L}^d$ for every $t \in [0, T]$ and for some $C > 0$.

Theorem ([Di Perna-Lions '89], [Ambrosio '04])

Let us assume that $|\nabla \mathbf{b}_t| \in L^1_{loc}(\mathbb{R}^d)$, $\operatorname{div} \mathbf{b}_t \in L^\infty(\mathbb{R}^d)$ *and*

$$\frac{|\mathbf{b}_t(x)|}{1 + |x|} \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d).$$

Then there exists a unique regular Lagrangian flow $\mathbf{X}$ *of* $\mathbf{b}$.

Remarks on the DiPerna-Lions theorem

The regularity assumption $|\nabla \mathbf{b}_t| \in L^1_{loc}(\mathbb{R}^d)$ can be replaced by

- $\nabla \mathbf{b}_t$ is a matrix-valued finite measure , [Ambrosio 04];
- singular integrals of L^1 functions, [Bohun, - Bouschut, Crippa 13].

The assumption $\operatorname{div} \mathbf{b}_t \in L^\infty(\mathbb{R}^d)$ can be weakened to $\operatorname{div} \mathbf{b}_t \in BMO(\mathbb{R}^d)$ [Mucha, 2010], [C., Crippa, Spirito 2016].

A different approach to this result was proposed by [Crippa, De Lellis, 08], considering a functionals of the type

$$\Phi_\delta(t) := \int \log\left(1 + \frac{|\mathbf{X}_1(t,x) - \mathbf{X}_2(t,x)|}{\delta}\right) dx \qquad t \in [0,T];$$

Question: a.e. uniqueness of integral curves

Does any divergence free $\mathbf{b} \in L^1_t W^{1,p}_x$ admit a unique integral curve (namely, $\gamma \in W^{1,1}(0,T)$ solution of the ODE $\dot\gamma(t) = \mathbf{b}(t,\gamma)$) for a.e. initial datum $x \in \mathbb{R}^d$?

Open since the pioneering works of DiPerna-Lions and Ambrosio.

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Main result

If $p < d$ then the a.e. uniqueness for trajectories does not hold.

Theorem ([Brué-C.-DeLellis, '20])

For every $d \geq 2$, $p < d$ and $s < \infty$ there exist a divergence free velocity field $\mathbf{b} \in C_t(W_x^{1,p} \cap L_x^s)$ and a set $A \subset \mathbb{T}^d$ such that

- $\mathscr{L}^d(A) > 0$;
- *for any $x \in A$ there are at least two integral curves of $\mathbf{b}$ starting at x.*

[Sorella, Pitcho, '21] and [Sorella, Giri, '21] show that the set A can be taken of full measure in the torus and that the theorem adapts to hamiltonian structures.

What about the critical case $p = d$?

Our strategy

Ingredients of proof:

- Ambrosio's superposition principle to connect the a.e. uniqueness of trajectories to uniqueness results for positive solutions to (CE).
- A new (asymmetric) Lusin-Lipschitz type inequality.
- Non-uniqueness theorem for positive solutions to (CE) based on convex integration type techniques borrowed from [Modena-Székelyhidi '18].

Superposition solutions - informally

Take a vector field **b** with two different flows. Then we observed that both

$$\mathbf{X}_1(t,\cdot)_\# \mu_0 \qquad \text{and} \qquad \mathbf{X}_2(t,\cdot)_\# \mu_0$$

solve the CE starting from μ_0. By linearity,

$$\lambda \mathbf{X}_1(t,\cdot)_\# \mu_0 + (1-\lambda)\mathbf{X}_2(t,\cdot)_\# \mu_0$$

is a solution as well. We can interpret this as "choosing $\mathbf{X}_1$ with probability λ and $\mathbf{X}_2$ with probability $1-\lambda$".

Ambrosio's superposition principle

A measure valued solution $\mu \in L^\infty_t(\mathscr{M}_+)$ to (CE) with velocity $\mathbf{b}$ is a superposition solution if for μ_0-a.e. $x \in \mathbb{T}^d$ there exists $\eta_x \in \mathscr{P}(C([0,T],\mathbb{T}^d))$ such that

- η_x is concentrated on integral curves of $\mathbf{b}$ starting at x;
- we have the representation formula $\mu = (e_t)_\#(\mu_0 \otimes \eta_x)$,

$$\int \phi \, d\mu_t = \int \left(\int \phi(\gamma(t)) \, d\eta_x(\gamma) \right) d\mu_0(x).$$

Superposition solutions are averages of integral curves of u.

Theorem ([Ambrosio '04])

Let $\mathbf{b} : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, $\mu \in L^\infty_t(\mathscr{M}_+)$ *solution of CE with*

$$\int_0^T \int |\mathbf{b}(t,x)| \, d\mu_t(x) \, dt < \infty.$$

Then it is a superposition solution.

Nonuniqueness by convex integration

If we produce an example of nonuniqueness of positive solutions of the continuity equation in some range of exponents we have disproved the a.e. uniqueness of integral curves.

Theorem ([Brué-C.-DeLellis, '20])

Let $d \geq 2$, $p \in (1, \infty)$, $r \in [1, \infty]$, $\frac{1}{r} + \frac{1}{r'} = 1$ be such that

$$\frac{1}{p} + \frac{1}{r} > 1 + \frac{1}{d}.$$

Then there exist $(\mathbf{b}, u)$ solution of the CE with

- *a divergence-free vector field $\mathbf{b} \in C_t(W^{1,p}_x \cap L^{r'}_x)$,*
- *a positive, nonconstant $u \in C_t L^r_x$ with $u(0, \cdot) = 1$*

Remarks

- The main theorem follows: any velocity field obtained in the previous theorem does not have the a.e. uniqueness for integral curves. Indeed
  - Since $\operatorname{div} \mathbf{b} = 0$, the function $\bar{u} \equiv 1$ solves CE.
  - The u constructed in this theorem is a second distinct solution!
  - As seen before, a.e. uniqueness of integral curves implies uniqueness of positive solutions to (CE).
- The construction is based on convex integration scheme, as in the groundbreaking works [DeLellis-Székelyhidi, '09-'13], [Isett '16] for the Euler equation and [Buckmaster-Vicol '17] for Navier-Stokes.
- The first ill-posedness result for (CE) with Sobolev velocity field has been proven in [Modena-Székelyhidi, '18], [Modena-Sattig, '19].
- Main novelties: positive solutions, a simpler convex integration scheme in any dimension.

The convex integration scheme

- We start from CE solved with an error

$$\begin{cases} \partial_t u_q + \mathrm{div}\,(\mathbf{b}_q u_q) = \mathrm{div}\, R_q \\ \mathrm{div}\,\mathbf{b}_q = 0 \end{cases}$$

  Solutions are obtained through an inductive procedure as $u = \lim_{q\to\infty} u_q$, $u = \lim_{q\to\infty} \mathbf{b}_q$ and $\lim_{q\to\infty} \|R_q\|_{L^1} = 0$.

- We look for $\mathbf{b}_{q+1} = \mathbf{b}_q + aB_{q+1}$, $u_{q+1} = u_q + bU_{q+1}$, where B_q and U_q are "highly oscillating" time-dependent versions of Mikado-flows (Cf. [Daneri-Székelyhidi '17]). a and b are "slow" functions. They cancel the error when interact

$$|R_q - abB_q U_q| \ll 1.$$

- We exploit the scaling invariances of the equation by making B_q and U_q concentrated ([Buckmaster-Vicol, '17]).
  Heuristic idea: Ill-posedness happens when u "concentrates" where $\mathbf{b}$ is far from being Lipschitz (i.e. $\nabla\mathbf{b}$ is "big").

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The end

Thank you for your attention! *

* and thanks to D. Strütt, EPFL, for the first pictures and animation