New families of surfaces with canonical map of high degree

joint work (very much in progress) with F. Fallucca (Trento)

Roberto Pignatelli

Roberto.Pignatelli@unitn.it

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Overview

1. Beauville's Theorem
2. Product-quotient surfaces and canonical subgroups
3. Some infinite families
4. Search of (sporadic) examples

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Beauville's Theorem

Theorem (Beauville[a])

[a] Beauville, Arnaud *L'application canonique pour les surfaces de type général.* (French) Invent. Math. **55** (1979), no. 2, 121–140.

Let S be a (compact complex) surface whose canonical image $\Sigma := \varphi_K(S) \subset \mathbb{P}^{p_g(S)-1}$ is a surface. Then

1. *either $p_g(\Sigma) = 0$*
2. *Σ is a canonical surface[a].*

[a] This means that Σ is embedded by its canonical map. In other words, the canonical map of any smooth surface birational to Σ is the (birational) map to Σ

It is easy to obtain examples of both phenomena with canonical map of minimal degree d, $d = 2$ in case 1 and $d = 1$ in case 2.

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Maximal degrees[1]

- If $p_g(\Sigma) = 0$ then $d \leq 36$ (Persson).
  The bound is sharp (Rito).
- If Σ is a canonical surface then $d \leq 9$.
  the current record is 5 (Pardini).

Up to finitely many families (Beauville)

- If $p_g(\Sigma) = 0$ then $d \leq 8$ (Xiao Gang)
  The bound is sharp (Beauville).
- If Σ is a canonical surface then $d \leq 3$
  we know infinitely many families with $d = 2$ but not with $d = 3$

Contributions by Beauville, Bin Nguyen, Catanese, Ciliberto, Gleissner, Mendes Lopes, Pardini, Persson, P-, Rito, Tovena, Xiao Gang...

Several open questions.

[1] Mendes Lopes, Margarida; Pardini, Rita, *On the degree of the canonical map of a surface of general type*, arxiv:2103.01912

What are product-quotient surfaces?

Definition

We consider two curves C_1 and C_2 of genus at least 2, an abstract finite group G and two monomorphisms $G \subset \mathrm{Aut}(C_j)$.
The induced product-quotient surface is the minimal resolution of the singularities S of the quotient $\frac{C_1 \times C_2}{G}$ where G acts as $g(x,y) = (gx, \varphi(g)y)$ for some $\varphi \in Aut(G)$.

We know

$$p_g(S) = p_g\left(\frac{C_1 \times C_2}{G}\right) = h^0(K_{C_1 \times C_2})^G$$

More precisely the pull-back gives an isomorphism

$$H^0(K_S) \cong H^0(K_{\frac{C_1 \times C_2}{G}}) \cong H^0(K_{C_1 \times C_2})^G \tag{1}$$

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Can product-quotient surfaces help?

$$H^0(K_S) \cong H^0(K_{\frac{C_1 \times C_2}{G}}) \cong H^0(K_{C_1 \times C_2})^G \qquad (2)$$

Carlos Rito suggested to consider the following situation.

Definition

In the above situation we say that $H \subset G$ is a **canonical subgroup** if $p_g\left(\frac{C_1 \times C_2}{H}\right) = p_g\left(\frac{C_1 \times C_2}{G}\right)$. In other words $H^0(K_{C_1 \times C_2})^H = H^0(K_{C_1 \times C_2})^G$.

By (2), then the canonical map of $\frac{C_1 \times C_2}{H}$ factors through the one of $\frac{C_1 \times C_2}{G}$ and therefore, if their canonical image is a surface, the degree of the canonical map of $\frac{C_1 \times C_2}{H}$ is a multiple of the index $[G : H]$.

So we look for canonical subgroups.

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First try (2019)

Carlos Rito and Christian Gleissner (unpublished) studied some special cases with abelian G producing several examples of canonical subgroups of index 2 and a couple of index 3.

I tried then to adapt to generalize their work to general finite groups, but the program appeared computationally not feasible.

The computionally hard part, essentially, is in managing the "Hurwitz moves".

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Some representation theory

In 2019 I did not consider the action of G on $H^0(K_{C_1 \times C_2})$: the geometric genera of the surfaces $\frac{C_1 \times C_2}{G}$ and $\frac{C_1 \times C_2}{H}$ are computed by M. Noether formula.

For a product-quotient surface it is possible to compute explicitly the action of G on $H^0(K_{C_1 \times C_2})$ by the Chevalley-Weil formula.

With Federico Fallucca we tried to understand condition

$$H^0(K_{C_1 \times C_2})^H = H^0(K_{C_1 \times C_2})^G \tag{3}$$

starting from the simplest possible case

$$G \cong \mathbb{Z}/2p\mathbb{Z} \qquad\qquad [G : H] = 2$$

for p prime.

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Wiman curves

Fon any odd number p, we consider the curve[2]

$$W_p := \{y^2 = (x_0^p - x_1^p)x_1\} \subset \mathbb{P}\left(1, 1, \frac{p+1}{2}\right)$$

It is a hyperelliptic curve of genus $\frac{p-1}{2}$ with an automorphism of order $2p$ generated by

$$(x_0, x_1, y) \mapsto (e^{\frac{2\pi i}{p}} x_0, x_1, -y)$$

The quotient gives a *triangle curve* $W_p \to \mathbb{P}^1$ branched at $\{0, 1, \infty\}$ with *signature*[3] $(2, p, 2p)$ corresponding respectively to the *small orbits* $\{y = 0, x_1 \neq 0\}$, $\{x_0 = 0\}$, $\{x_1 = 0\}$.

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[2] Anders Wiman, *Ueber die hyperelliptichen Kurven und diejenigen vom Geschlecht $p = 3$, welche eindeutige Transformationen in sich selbst zulassen*, Bihang. Till. Kungl. Svenska Vetenskaps-Akademiens Handlingar 21, no. 1 (1895), 1-23.

[3] A spherical system of generators is $([1]_{2p}, [p-1]_{2p}, [p]_{2p})$

Generalized Wiman curves

Definition

A **generalized Wiman curve** $(C, \mathbb{Z}/2p\mathbb{Z})$ is a cyclic cover of $\mathbb{P}^1$ of order $2p$, p odd, with signature $(2^m, a, b)$

Then $a, b \in \{p, 2p\}$. Either m is odd and $a \neq b$ or m is even $a = b$.

For each choice of $p, m \geq 1$, p odd, and $a, b \in \{p, 2p\}$ as above there is exactly one generalized Wiman curve up to automorphisms.

For $m = 1$ we get W_p.

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Generalized Wiman product-quotient surfaces

Definition

A **generalized Wiman product-quotient surface** is a product quotient-surface $\frac{C_1 \times C_2}{\mathbb{Z}/2p\mathbb{Z}}$ where $(C_j, \mathbb{Z}/2p\mathbb{Z})$ are generalized Wiman curves.

For all these surfaces the only subgroup of index 2 of $\mathbb{Z}/2p\mathbb{Z}$ is canonical.

This is a triply infinite family (depending on p, m_1, m_2 where m_j is the number of 2 in the signature relative to the curve C_j, we can also choose $a_i = b_i \in \{p, 2p\}$ if m_i is even and an automorphism of G).

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A theorem (?)

Theorem

Let p be an odd prime. Assume that $\frac{C_1 \times C_2}{\mathbb{Z}/2p\mathbb{Z}}$ is regular and it has a canonical subgroup of index 2. Then one of the following occur

GW) $\frac{C_1 \times C_2}{\mathbb{Z}/2p\mathbb{Z}}$ *is a generalized Wiman product-quotient surface*

W) $(C_1, \mathbb{Z}/2p\mathbb{Z}) = W_p$

F) *A very technical case, not giving surfaces of general type*[a] *at least up to* $p = 41$.

S) *Two sporadic examples with* $p = 3$ *whose quotient* $(C_1 \times C_2)/H$ *has* $(p_g, q, c_1^2) = (3, 1, 6)$ *and* $(6, 2, 24)$.

[a] but infinitely many surfaces of Kodaira dimension 1

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Case GW

In both cases *GW* and *W* the canonical map of $\frac{C_1 \times C_2}{\mathbb{Z}/2p\mathbb{Z}}$ factors through a *further* involution whose quotient is ruled.
We can prove:

Theorem

In case GW, for m_1, m_2 big enough, the canonical map of $(C_1 \times C_2)/H$ is of degree 4 on a ruled surface.

The only infinite family that I knew with canonical map of degree 4 is the product of hyperelliptic curves[4].
We obtain unbounded families with slope K^2/χ assuming infinitely many accumulation points in the range $[7, 8]$ (*e.g.* all $8 - \frac{1}{k}$ for $k \in \mathbb{N}$ not prime).

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[4]They have $K^2/\chi = 8$

A couple of databases

Jennifer Paulhus has constructed a database of pairs (C, G). *Currently the database contains all groups G acting as automorphisms of curves C of genus 2 to 15 such that* C/G *has genus* 0*, as well as genus 2 through 4 with quotient genus greater than* 0.

Different entries of the database may give isomorphic pairs (C, G) (she avoided dealing with the Hurwitz moves).

With Alessandro Ghigi and Diego Conti we found an efficient way to manage Hurwitz moves, and we are building a similar database (hopefully) up to genus 30-40.

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Some canonical subgroups of index 4

Federico run a systematic search of canonical subgroups using our preliminary database, assuming both curves of genus at most 10.

For $p_g = 3$ he has found examples with canonical subgroups of index up to 6.

For $p_g \geq 4$ he has found only canonical subgroups of index smaller than 4: exactly 11 distinct examples with canonical index 4.

Much work to be done here!

Representation Theory of Generalized Wiman curves

Definition

For a curve C with a group of automorphism G we denote by $\chi_C \colon G \to \mathbb{C}$ the character of the canonical representation of G on $H^0(C, K_C)$.

Then by Chevalley-Weil formula we can write

$$\chi_{W_p} = \sum_{k \ odd, \ p<k<2p} \epsilon^k$$

where ϵ is a generator of $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$
In other words all eigenspaces of $H^0(C, K_C)$ have dimension 1, with "eigenvalues" of exponent odd and bigger than p.
For generalized Wiman curves we obtain $\chi_C = \sum_{k \ odd} a_k \epsilon^k$, $a_k \in \mathbb{N}$.

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Representation Theory of Generalized Wiman Product-Quotient surfaces

By the Künneth formula, if we consider a diagonal action of $\mathbb{Z}/2p\mathbb{Z}$ (shifting the action on the second factor by any automorphism φ) on the product of two generalized Wiman curves C_1 and C_2, then the character of the representation on

$$H^0(K_{C_1 \times C_2}) \cong H^0(K_{C_1}) \otimes H^0(K_{C_2})$$

is of the form[5] $\sum_{k \text{ even}} b_k \epsilon^k$, $b_k \in \mathbb{N}$.
Since p is odd, ϵ^p do not appear: that's condition (3).

Since also the other odd exponents do not appear, then $H^0(K_{C_1 \times C_2})$ is invariant by the involution $(p, 0)$, whose quotient is ruled.

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[5] the sum of two odd numbers is even

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