On a dimension-free control on L2 of the Riesz transform in terms of the Riesz transform

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Let $R_j f(x) = \lim_{t \to 0} R_j^t f(x)$, $j = 1, \dots, d$, with

$$R_j^t f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \int_{|x-y|>t} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy$$

(Truncated Riesz transform at level $t > 0$)

We will be intersted in

$$R_j^* f(x) = \sup_{t>0} \left| R_j^t f(x) \right|$$

(Maximal truncated Riesz transform)

and its relation with R_j

It is well known that

$$\| R_j^* (f) \|_{L^p(\mathbb{R}^d)} \leq C(p,d) \|f\|_{L^p(\mathbb{R}^d)}, \qquad 1<p<\infty$$

This follows e.g. from

$$R_j^* (f) \leq C(d) \left(\mathcal{M}(R_j f) + \mathcal{M}(f) \right)$$ (Cotlar's inequality),

where $\mathcal{M}$ is the (centered) Hardy-Littlewood maximal function

In 2006 Mateu and Verdera proved that

$$R_j^*(f)(x) \leq C(d)\,(M \circ M)(R_j f)(x)$$

[Mateu–Verdera 2006]

$$\Downarrow$$

$$\|R_j^*(f)\|_{L^p(\mathbb{R}^d)} \leq C(p,d)\,\|R_j f\|_{L^p(\mathbb{R}^d)}, \quad 1<p<\infty$$

The constant in the first inequality above is $C(d) \sim 4^d$

We prove that

Theorem (Kucharski–Wróbel 2021)

$$\|R_j^* f\|_{L^2(\mathbb{R}^d)} \leq 2 \cdot 10^8\,\|R_j f\|_{L^2(\mathbb{R}^d)}$$

Corollary

$$\Big\| \Big(\sum_{j=1}^{d} |R_j^*(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^d)} \leq 2 \cdot 10^8 \, \|f\|_{L^2(\mathbb{R}^d)}$$

Possible approaches and problems

a) Cotlar's inequality

→ extra term $M(f)$

→ dimensional dependence

b) Method of rotations cf. [Duoandikoetxea - Rubio de Francia 1985]

→ dimension independent ✓ but

→ control of the form $\|R_j^* f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$

Our approach – STEP 1 – FACTORISATION

$$m(u) = \frac{2^{\frac{d+1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} \int_{2\pi u}^{\infty} r^{-\frac{d}{2}} J_{\frac{d}{2}}(r)\,dr, \quad u > 0$$

$$M^t f(x) = \mathcal{F}^{-1}\left(m(t\cdot|\xi|)\,\mathcal{F}f\right)(x), \quad x \in \mathbb{R}^d$$

Lemma (Factorisation)

$$R_j^t f = M^t (R_j f), \quad f \in \mathcal{S}(\mathbb{R}^d)$$

Consequently the optimal constant $\mathcal{O}(p,d)$ in

$$\|R_j^* f\|_{L^p(\mathbb{R}^d)} \leq \mathcal{O}(p,d)\,\|R_j f\|_{L^p(\mathbb{R}^d)} \quad \text{equals} \quad \|M^*\|_{L^p(\mathbb{R}^d)}$$

Why is the factorisation possible ?? Consider $t=1$

Then $R_j^1 f = f * K_j^1$ with $K_j^1 = c_d \, x_j \, K(x)$,

where $K(x) = |x|^{-d-1} \chi_{|x|>1}$

Thus

$$(K_j^1)^\wedge(\xi) = c_d \, (x_j K)^\wedge(\xi) = -\frac{c_d}{2\pi i} \partial_j \widehat{K}(\xi)$$

But K is radial thus $\widehat{K}(\xi) = h(|\xi|)$, hence

$$(K_j^1)^\wedge(\xi) = \tilde{c}_d \, \frac{\xi_j}{|\xi|} \, h'(|\xi|)$$

Comments on the Factorization

→ at the kernel level $R_j^t f = h_t * (R_j f)$ close to [Mateu-Verdera]

→ the kernel is not non-negative

→ not easy to work with the kernel

for dimension-free estimates

→ By (Factorization) it is enough to prove

Theorem' $M^* f = \sup_{t>0} |M^t f|$ satisfies

$$\| M^* f \|_{L^2(\mathbb{R}^d)} \leq 2 \cdot 10^8 \, \|f\|_{L^2(\mathbb{R}^d)}$$

How to prove Theorem'

→ based on ideas developed by [Bourgain 86'] for proving dim-free estimates of HL maximal operators over convex sets.

→ and a more recent interpretation of these methods by [Bourgain, Mirek, Stein, W 18'] and [Mirek, Stein, Zorin-Kranich 20']

→ Crucial ingredient → Fourier transform estimates

How are the Fourier transform estimates useful?

We have

$$\sup_{t>0} |M^t f| \leq \sup_n |M^{2^n} f| + \Big(\sum_n \sup_{t \in [2^n, 2^{n+1})} |M^t f - M^{2^n} f|^2 \Big)^{\frac{1}{2}}$$

$$\leq \underbrace{\sup_n |P^{2^n} f|}_{\text{Poisson semigroup}} + \underbrace{\Big(\sum_n |M^{2^n} f - P^{2^n} f|^2 \Big)^{\frac{1}{2}}}_{:= L(f)} + \underbrace{\Big(\sum_n \sup_{t \in [2^n, 2^{n+1})} |M^t f - M^{2^n} f|^2 \Big)^{\frac{1}{2}}}_{: S(f)}$$

STEP 2 - DIM-FREE FOURIER TRANSFORM ESTIMATES

Lemma (FT estimates) There is a universal $C>0$ such that

1) $|m(u)-1| \leq C \frac{u}{\sqrt{d}}$

2) $|m(u)| \leq C \left(\frac{u}{\sqrt{d}}\right)^{-1}$, $\quad u>0$

3) $|u\, m'(u)| \leq C$

Remarks:

→ 1) and 2) are enough for $L(f) = \left(\sum_n |M^{2^n} f - P^{2^n} f|^2\right)^{\frac{1}{2}}$

→ for obtaining dim-free estimates it is crucial that we have $\frac{u}{\sqrt{d}}$ in 1) and $\left(\frac{u}{\sqrt{d}}\right)^{-1}$ in 2)

A flavour of the proof of the (FT estimates) $\to$ 3)

Since $m(u) = \frac{2^{\frac{d+1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} \int_{2\pi u}^{\infty} r^{-\frac{d}{2}} J_{\frac{d}{2}}(r)\,dr$ we have

$$u\, m'(u) = \frac{-2^{\frac{d+1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} \left(2\pi u\right)^{-\frac{d}{2}} J_{\frac{d}{2}}(2\pi u) \cdot 2\pi$$

We need estimates for the Bessel function $J_{\frac{d}{2}}$ independent of d.

Recall $$J_{\frac{d}{2}}(2\pi u) = \frac{(2\pi u)^{\frac{d}{2}}}{2^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} e^{i 2\pi u s} \left(1-s^2\right)^{\frac{d}{2}-\frac{1}{2}} ds$$

Lemma (Bessel function estimates) [M. MIREK – T. Z. SZAREK – BW 2020]
[M. KUCHARSKI – BW 2021]

For $u>0$ we have

$$\left| J_{\frac{d}{2}}(2\pi u) \right| \leq 2100 \cdot \frac{(2\pi u)^{\frac{d}{2}}}{2^{\frac{d}{2}} \Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\sqrt{\pi}} \cdot \sqrt{\frac{2}{d}} \left(e^{-\frac{2\pi u}{\sqrt{d}}} + e^{-\frac{d}{10}} \right),$$

Thus, for $u>0$

$$|u\, m'(u)| \leq 8400 \cdot \frac{u}{\sqrt{d}} \left(e^{-\frac{2\pi u}{\sqrt{d}}} + e^{-\frac{d}{10}} \right)$$

Remarks

– The above estimate is useful for e.g. $u<d$ and implies

$$|u\, m'(u)| \leq C$$

Remarks ctd.

- For $u \geq d$ we just use $|J_{\frac{d}{2}}(u)| \leq 1$ which gives

$$\left| u\, m'(u) \right| \leq C \, \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d}{2}}} \, u^{-\frac{d}{2}+1} \leq C \left(\frac{d+1}{2}\right)^{\frac{d+1}{2}} d^{-\frac{d}{2}+1} \leq C$$

- the Bessel function estimates are proved by a change of contour of integration in

$$J_{\frac{d}{2}}(2\pi u) = \frac{(2\pi u)^{\frac{d}{2}}}{2^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} e^{i2\pi u s} (1-s^2)^{\frac{d}{2}-\frac{1}{2}} \, ds$$

$$= \frac{(2\pi u)^{\frac{d}{2}}}{2^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\sqrt{\pi}} \cdot \sqrt{\frac{2}{d}} \cdot \int_{-\sqrt{\frac{d}{2}}}^{\sqrt{\frac{d}{2}}} e^{i2\pi \frac{\sqrt{2}u}{\sqrt{d}} s} \left(1 - \frac{2s^2}{d}\right)^{\frac{d}{2}-\frac{1}{2}} ds$$

Thank you for your attention!

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Greetings from Portorož