Balanced Hermitian metrics

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Balanced metrics

Definition (Michelsohn)

A balanced metric on a n-dim complex manifold is an Hermitian metric ω such that $d(\omega^{n-1}) = 0$.

• A metric is balanced if and only if $\Delta_{\partial} f = \Delta_{\overline{\partial}} f = 2\Delta_d f$ for every $f \in \mathcal{C}^{\infty}(M, \mathbb{C})$ (Gauduchon).

• A compact complex manifold M admits a balanced metric if and only if M carries no positive currents of degree $(1, 1)$ which are components of a boundary (Michelsohn).

In particular, Calabi-Eckmann manifolds have no balanced metrics!

Examples of balanced manifolds

- The twistor space of a 4-dim oriented anti-self-dual Riemannian manifold always has a balanced metric (Michelsohn; Gauduchon).
- Every compact complex manifold bimeromorphic to a compact Kähler manifold is balanced (Alessandrini, Bassanelli) $\Rightarrow$

Moishezon manifolds and complex manifolds in the Fujiki class $\mathcal{C}$ are balanced.

- A class of non-Kähler balanced manifolds costructed by using conifold transictions which includes $\#_k(S^3 \times S^3)$, $k \geq 2$ [Li, Fu, Yau].
- Any left-invariant Hermitian metric on a unimodular complex Lie group is balanced [Abbena, Grassi].
- A characterization of compact complex homogeneous spaces with invariant volume admitting a balanced metric (in particular $c_1 \neq 0$) [F, Grantcharov, Vezzoni].

Classification results on Lie groups

• 6-dim balanced nilpotent Lie algebras [Ugarte].

• 6-dim balanced unimodular solvable Lie algebras admitting a holomorphic $(3,0)$-form [F, Otal, Ugarte].

Problem

Classify balanced almost abelian Lie algebras $\mathfrak{g}$ (i.e. with abelian ideal $\mathfrak{h}$ of codimension one).

$\hookrightarrow$ $\mathfrak{g} = \mathbb{R} \ltimes_B \mathfrak{h}$, with $B \in \mathrm{End}(\mathfrak{h})$.

We can use the characterization of Hermitian almost abelian Lie algebras [Lauret, Rodriguez-Valencia; Arroyo, Lafuente].

Let $\mathfrak{g} = \mathbb{R} \ltimes_B \mathfrak{h}$ be a $2n$-dim almost abelian Lie algebra.

If $\mathfrak{g}$ admits a Hermitian structure (J, g), then $\exists$ a ON basis (e_i) s.t.

$$\mathfrak{h} = \text{span} < e_1, \dots, e_{2n-1} >, \ \mathfrak{h}^\perp = \text{span} < e_{2n} >,$$
$$\mathfrak{h}_1 := \mathfrak{h} \cap J\mathfrak{h} = \text{span} < e_2, \dots, e_{2n-1} >,$$
$$Je_1 = e_{2n}, \ Je_i = e_{2n+1-i}, i = 1, \dots, n,$$
$$ad_{e_{2n}}|_{\mathfrak{h}} = \begin{pmatrix} a & 0 \\ v & A \end{pmatrix}, \ a \in \mathbb{R}, \ v \in \mathfrak{h}_1, \ A \in \mathfrak{gl}(\mathfrak{h}_1), \ [A, J] = 0.$$

Proposition (F, Paradiso)

(J, g) is balanced $\iff$ $v = 0$, $\text{tr}(A) = 0$.

$\hookrightarrow$ 9 isomorphism classes in dim 6.

Interplay with other types of Hermitian metrics

A Hermitian metric which is balanced and puriclosed is Kähler [Alexandrov, Ivanov; Popovici].

Conjecture

Every compact complex manifold admitting a balanced and a pluriclosed metric is Kähler.

The conjecture is true for all the known examples of compact balanced manifolds!

Theorem (F, Grantcharov, Vezzoni)

There exists a compact complex non-Kähler manifold admitting a balanced and an astheno-Kähler metric.

$\hookrightarrow$ negative answer to a question posed by Székeleyhidi, Tosatti, Weinkove.

Balanced flow

Let (M^{2n}, J, ω_0) be a complex manifold with a balanced metric ω_0.

Definition (Bedulli, Vezzoni)

A parabolic flow preserving the balanced condition is given by:

$$\partial_t \varphi(t) = i\partial\overline{\partial} *_t (\rho^C_{\omega(t)} \wedge *_t \varphi(t)) + \Delta_{BC}\varphi(t), \quad \varphi(0) = *_0\omega_0,$$

where $\rho^C_{\omega(t)}$ is the Ricci form of the Chern connection and

$$\Delta_{BC} = \partial\overline{\partial}\overline{\partial}^*\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \overline{\partial}^*\partial\partial^*\overline{\partial} + \partial^*\overline{\partial}\overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} + \partial^*\partial$$

is the Bott-Chern Laplacian.

Short-time existence and uniqueness for compact manifolds [Bedulli, Vezzoni].

Remark

If ω_0 is Kähler, then the flow coincides with the Calabi flow:

$$\begin{cases} \partial_t \omega(t) = i\partial\overline{\partial} s_{\omega(t)}, & \omega(t) \in \{\omega_0 + i\partial\overline{\partial} u > 0\} \subset [\omega_0] \\ \omega(0) = \omega(0), & \end{cases}$$

where $s_{\omega(t)}$ is the scalar curvature of $\omega(t)$.

Problem

Study the Balanced flow on almost abelian Lie groups (G, J, g).

We use the bracket flow introduced by Lauret, i.e. we evolve the Lie bracket instead of the Hermitian metric g!

Choose $(J_0, \langle \cdot, \cdot \rangle)$ linear Hermitian structure on $\mathbb{R}^{2n}$.

Fix a basis (e_i) making $(\mathfrak{g}, J, g_0) \cong (\mathbb{R}^{2n}, J_0, \langle \cdot, \cdot \rangle)$.

$\hookrightarrow$ The Lie bracket $\mu(t) = \mu(a(t), v(t), A(t)) \in \Lambda^2(\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$ evolves as

$$a' = p\, a, \;\; v' = 0, \;\; A' = [A, P] + p\, A,$$

where $p := p(a, A)$ and $P := P(a, A)$ are fourth-order polynomials.

Theorem (F, Paradiso)

Let (G, J, ω_0) be a 6-dim balanced almost abelian Lie group. Then

- *the solution $\omega(t)$ to the balanced flow is defined for all positive times (eternal solution);*
- *Cheeger-Gromov convergence to a Kähler almost abelian Lie group.*

Hull-Strominger System

Remark

It describes the geometry of compactification of heterotic superstrings with torsion to 4-dimensional Minkowski spacetime.

• M be a compact 3-dim complex manifold with a nowhere vanishing holomorphic $(3,0)$-form Ω.

• E be a complex vector bundle over M with a Hermitian metric H along its fibers and let $\alpha' \in \mathbb{R}$ be a constant (slope parameter).

The Hull-Strominger system, for the Hermitian metric ω on M, is:

(1) $F_H^{2,0} = F_H^{0,2} = 0$, $F_H \wedge \omega^2 = 0$ (Hermitian-Yang-Mills),

(2) $d(\|\Omega\|_\omega \, \omega^2) = 0$ (ω is conformally balanced),

(3) $i\partial\bar{\partial}\omega = \frac{\alpha'}{4}(Tr(R_\nabla \wedge R_\nabla) - Tr(F_H \wedge F_H))$ (Bianchi identity)

where F_H, R_∇ are the curvatures of H and of a metric connection ∇ on TM.

Link with balanced metrics

The 2nd equation $d(\|\Omega\|_\omega \omega^2) = 0$ says that ω is conformally balanced.

Remark

It was originally written as $d^*\omega = i(\bar{\partial} - \partial)\ln(\|\Omega\|_\omega)$ (the equivalence was proved by Li and Yau).

The Hull-Strominger system can be interpreted as a notion of "canonical metric" for conformally balanced manifolds.

Remark

$F_H^{2,0} = F_H^{0,2} = 0$, $F_H \wedge \omega^2 = 0$ is the Hermitian-Yang-Mills equation which is equivalent to E being a stable bundle.

Remark

• Calabi-Yau manifolds can be viewed as special solutions (with $E = T^{1,0}M$ and $H = \omega$) [Candelas, Horowitz, Strominger, Witten].

• Since ω may not be Kähler, there is a one-parameter line ∇^τ of natural unitary connections on $T^{1,0}M$ defined by ω, passing through the Chern connection ∇^C and the Bismut connection ∇^B.

For $\nabla = \nabla^C$ the first Non-Kähler solutions have been found by Fu and Yau on a class of toric fibrations over K3 surfaces, constructed by Goldstein and Prokushkin.

Main Idea: reduce the Hull-Strominger system to a 2-dimensional Monge-Ampère equation.

The construction of Goldstein and Prokushkin

Let (S, ω_S) be a K3 surface with Ricci flat Kähler metric ω_S.

- To any pair ω_1, ω_2 of anti-self-dual (1,1)-forms on S such that $[\omega_i] \in H^2(S, \mathbb{Z})$, Goldstein and Prokushkin associated a toric fibration

$$\pi : M \to S,$$

with a nowhere vanishing holomorphic 3-form $\Omega = \theta \wedge \pi^*(\Omega_S)$, for a (1,0)-form $\theta = \theta_1 + i\theta_2$, where θ_i are connection 1-forms on M such that $d\theta_i = \pi^* \omega_i$.

- The $(1, 1)$-form

$$\omega_0 = \pi^*(\omega_S) + i\theta \wedge \overline{\theta}$$

is a balanced Hermitian metric on M, i.e. $d\omega_0^2 = 0$.

The Fu -Yau solution

Fu and Yau found a solution of the Hull-Strominger system with M given by the Goldstein-Prokushkin construction, and the following ansatz for the metric on M:

$$\omega_u = \pi^*(e^u \omega_S) + i\theta \wedge \overline{\theta},$$

where u is a function on S. This reduces the Hull-Strominger system to a 2-dim Monge-Ampère equation with gradient terms:

$$i\partial\overline{\partial}(e^u - fe^{-u}) \wedge \omega + \alpha' i\partial\overline{\partial}u \wedge i\partial\overline{\partial}u + \mu = 0,$$

under the ellipticity condition

$$(e^u + fe^{-u})\omega + 4\alpha' i\partial\overline{\partial}u > 0,$$

where $f \geq 0$ is a known function, and μ is a (2, 2)-form with average 0.

New solutions to Hull-Strominger System

Using that the argument by Fu and Yau depends only on the foliated structure of the 6-manifold, we show that the Fu-Yau solution on torus bundles over K3 surfaces can be generalized to torus bundles over K3 orbifolds $\hookrightarrow$

Theorem (F, Grantcharov, Vezzoni)

Let $13 \leq k \leq 22$ and $14 \leq r \leq 22$. Then on the smooth manifolds $S^1 \times \#_k(S^2 \times S^3)$ and $\#_r(S^2 \times S^4)\#_{r+1}(S^3 \times S^3)$ there are complex structures with trivial canonical bundle admitting a balanced metric and a solution to the Hull-Strominger system via the Fu-Yau ansatz.

The cases $k = 22$ and $r = 22$ correspond to Fu-Yau solutions.

To construct the explicit examples we consider T^2-bundles over an orbifold S which are given by the following sequence

$$\begin{array}{ccc} S^1 & \hookrightarrow & M \\ & & \downarrow \\ S^1 & \hookrightarrow & M_1 \\ & & \downarrow \\ & & S \end{array}$$

where $M_1 \to S$ is a Seifert S^1-bundle, M_1 is smooth and $M \to M_1$ is a regular principal S^1-bundle over M_1.

The Anomaly flow

The solutions of the Hull-Strominger system can be viewed as stationary points of the following flow of positive $(2,2)$-forms, called the "Anomaly flow"

$$\begin{cases} \partial_t(\|\Omega\|_{\omega(t)}\omega(t)^2) = i\partial\overline{\partial}\omega(t) + \alpha'(Tr(R_t \wedge R_t) - Tr(F_t \wedge F_t)) \\ H(t)^{-1}\partial_t H(t) = \dfrac{\omega(t)^2 \wedge F_t}{\omega(t)^3}, \quad \omega(0) = \omega_0,\ F(0) = F_0. \end{cases}$$

with ω_0 (conformally balanced) [Phong, Picard, Zhang].

In the compact case:

• Short-time existence and uniqueness [Phong, Picard, Zhang].

• For $t \to \infty$ the limit solves the Hull-Strominger system $\hookrightarrow$ new proof of Fu-Yau non-Kähler solutions [Phong, Picard, Zhang].

Flat case

Assume $F_t = 0$ for all t (i.e. E is flat)

$$\begin{cases} \partial_t(\|\Omega\|_{\omega(t)}\omega(t)^2) = i\partial\overline{\partial}\omega(t) + \alpha'(Tr(R_t^\tau \wedge R_t^\tau)) \\ \omega(0) = \omega_0, \end{cases} \quad (*)$$

where R^τ is the curvature of the Gauduchon connection ∇^τ, $\tau \in \mathbb{R}$ (for $\tau = 1$, $\nabla^\tau = \nabla^C$).

Theorem (F, Paradiso)

- *An almost abelian Lie algebra* $(\mathfrak{g}(a, v, A), J, g)$ *is balanced with a holomorphic* $(3,0)$*-form* $\iff a = 0, v = 0, tr(A) = tr(JA) = 0$.
- *The anomaly flow* $(*)$ *on almost abelian Lie groups preserves the balanced condition for every* $\tau, \alpha' \in \mathbb{R}$*, in the left-invariant case.*
- *Left-invariant locally conformal Kähler metrics on almost abelian Lie groups are fixed points of the anomaly flow* $(*)$.

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